

BIFURCATIONS FROM NONDEGENERATE FAMILIES OF PERIODIC SOLUTIONS

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ABSTRACT. By a nondegenerate k -parameterized family K of periodic solutions we understand the situation when the geometric multiplicity of the multiplier $+1$ of the linearized on K system equals to k . Bifurcation of asymptotically stable periodic solutions from K is well studied in the literature and different conditions have been proposed depending on whether the algebraic multiplicity of $+1$ is k or not (by Malkin, Loud, Melnikov, Yagasaki). In this paper we assume that the later is unknown. Asymptotic stability can not be understood in this case, but we demonstrate that the information about uniqueness of periodic solutions is still available. Moreover, we show that differentiability of the right hand sides is not necessary for the results of this kind and our theorems are proven under a kind of Lipschitz continuity.

1. INTRODUCTION

In [21] Malkin studied the bifurcation of T -periodic solutions in the n -dimensional T -periodic systems of the form

$$(1.1) \quad \dot{x} = f(t, x) + \varepsilon g(t, x, \varepsilon),$$

where both functions f and g are sufficiently smooth. It is assumed in [21] that the unperturbed system (namely (1.1) with $\varepsilon = 0$) has a family of T -periodic solutions, denoted $x(\cdot, \xi(h))$, whose initial conditions are given by a smooth function $\xi : \mathbb{R}^k \rightarrow \mathbb{R}^n$, $h \mapsto \xi(h)$. In these settings the adjoint linearized differential system

$$(1.2) \quad \dot{u} = -(Df(t, x(t, \xi(h))))^* u$$

has k linearly independent T -periodic solutions $u_1(\cdot, h), \dots, u_k(\cdot, h)$ and the geometric multiplicity of the multiplier $+1$ of (1.2) is, therefore, k . Assuming that the algebraic multiplicity of $+1$ is k either, Malkin proved [21] that if the bifurcation

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function

$$M(h) = \int_0^T \begin{pmatrix} \langle u_1(\tau, h), g(\tau, x(\tau, \xi(h)), 0) \rangle \\ \dots \\ \langle u_k(\tau, h), g(\tau, x(\tau, \xi(h)), 0) \rangle \end{pmatrix} d\tau$$

has a simple zero $h_0 \in \mathbb{R}^k$, then for any $\varepsilon > 0$ sufficiently small system (1.1) has a unique T -periodic solution x_ε such that $x_\varepsilon(0) \rightarrow \xi(h_0)$ as $\varepsilon \rightarrow 0$. Here simple zero means that $M(h_0) = 0$ and the Jacobian determinant of M at h_0 is nonzero. As usual $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbb{R}^n . Moreover, Malkin related the asymptotic stability of the solution x_ε with the eigenvalues of the Jacobian matrix $DM(h_0)$. The same result has been proved independently by Loud [20].

Malkin's result has been developed by Kopnin [15] and Loud [20] who studied the case when the zero h_0 of M is not necessary simple and both authors obtained conditions for the existence, uniqueness and asymptotic stability of the T -periodic solution x_ε of (1.1) satisfying $x_\varepsilon(0) \rightarrow \xi(h_0)$ as $\varepsilon \rightarrow 0$. Other improvements suppress the smoothness requirement for g and are due to Fečkan [11] and Kamenskii-Makarenkov-Nistri [13], but only the existence and the calculation of the topological index of solutions x_ε of (1.1) has been considered.

The analysis of the situation when the algebraic multiplicity of $+1$ is n goes back to Melnikov [22] and stability for simple and singular zeros h_0 of M has been achieved by Yagasaki [23].

In this paper (Theorems 2 and 3 below are our main results) we only assume that (1.2) has k linearly independent T -periodic solutions (i.e. the geometric multiplicity of the multiplier $+1$ of (1.2) is k). Also, we neither assume that the zero h_0 of M is simple, nor that g is differentiable. More precisely, we assume that in a small neighborhood of h_0 the topological degree of M is nonzero and that M is a so-called dilating mapping, while for g we assume that it is Lipschitz and "piecewise differentiable" in a suitable sense. Both assumptions are explicit and weaker than the Malkin's ones. Note that one of the conditions for g , denoted below by (A9), has its roots in [5, 6, 7]. On the other hand we do not obtain asymptotic stability of the T -periodic solution x_ε of (1.1) but we prove its existence and uniqueness, in particular we prove that it is isolated. In order to study the asymptotic stability one can eventually use a result of Kolesov's [14]. This result is a generalization of the Lyapunov linearization stability criterium for Lipschitz systems and it requires the isolateness of the T -periodic solution x_ε .

In order to prove our main result we need to generalize the Lyapunov-Schmidt reduction method (see [8]) for the case of nonsmooth finite dimensional functions. The application of the generalized Lyapunov-Schmidt reduction method for proving Theorem 2 is done by following the ideas of [4].

The paper is organized as follows. In the next section we summarize our notations. In Section 3 we generalize the Lyapunov-Schmidt reduction method for nonsmooth finite dimensional functions. In Section 4 we prove Theorem 2 and the main result of the paper, Theorem 3. An application of this theorem is done in Section 5.

2. NOTATIONS

The following notations will be used throughout this paper.

Let $n, m, k \in \mathbb{N}$, $k \leq n$, $i \in \mathbb{N} \cup \{0\}$.

We denote the projection onto the first k coordinates by $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$, and the one onto the last $n - k$ coordinates by $\pi^\perp : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$.

We denote by $I_{n \times n}$ the identity $n \times n$ matrix, while $0_{n \times m}$ denotes the null $n \times m$ matrix.

For an $n \times n$ matrix A we denote by A^* the adjoint of A , that in the case the matrix is real reduces to the transpose.

We consider a norm in \mathbb{R}^n denoted by $\|\cdot\|$. Let Ψ be an $n \times n$ real matrix. Then $\|\Psi\|$ denotes the operator norm, i.e. $\|\Psi\| = \sup_{\|\xi\|=1} \|\Psi\xi\|$.

Let $\xi \in \mathbb{R}^n$ and $\mathcal{Z} \subset \mathbb{R}^n$ compact, then we denote by $\rho(\xi, \mathcal{Z}) = \min_{\zeta \in \mathcal{Z}} \|\xi - \zeta\|$ the distance between ξ and \mathcal{Z} .

For $\delta > 0$ and $z \in \mathbb{R}^n$ the ball in \mathbb{R}^n centered in z of radius δ will be denoted by $B_\delta(z)$.

For a subset $\mathcal{U} \subset \mathbb{R}^n$ we denote by $\text{int}(\mathcal{U})$, $\overline{\mathcal{U}}$ and $\overline{\text{co}}\mathcal{U}$ its interior, closure and closure of the convex hull, respectively.

We denote by $C^i(\mathbb{R}^n, \mathbb{R}^m)$ the set of all continuous and i times continuously differentiable functions from \mathbb{R}^n into \mathbb{R}^m .

Let $\mathcal{F} \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ be a function that does not have zeros on the boundary of some open bounded set $\mathcal{U} \subset \mathbb{R}^n$. Then $d(\mathcal{F}, \mathcal{U})$ denotes the Brouwer topological degree of \mathcal{F} on \mathcal{U} (see [3] or [18, Ch. 1, § 3]).

For $\mathcal{F} \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $D\mathcal{F}$ denotes the Jacobian matrix of \mathcal{F} . If $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ and $\alpha \in \mathbb{R}^k, \beta \in \mathbb{R}^{n-k}$, then $D_\alpha \mathcal{F}(\cdot, \beta)$ denotes the Jacobian matrix of $\mathcal{F}(\cdot, \beta)$.

For $\mathcal{F} \in C^2(\mathbb{R}^n, \mathbb{R})$, $H\mathcal{F}$ denotes the Hessian matrix of \mathcal{F} , i.e. the Jacobian matrix of the gradient of \mathcal{F} .

Let $\delta > 0$ be sufficiently small. With $o(\delta)$ we denote a function of variable δ such that $o(\delta)/\delta \rightarrow 0$ as $\delta \rightarrow 0$, while $O(\delta)$ denotes a function of δ such that $O(\delta)/\delta$ is bounded as $\delta \rightarrow 0$. Here the functions o and O can depend also on other variables, but the above properties hold uniformly when these variables lie in a fixed bounded region.

We say that the function $Q : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is *locally uniformly Lipschitz with respect to its first variable* if for each compact $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there exists $L > 0$ such that $\|Q(z_1, \lambda) - Q(z_2, \lambda)\| \leq L\|z_1 - z_2\|$ for all $(z_1, \lambda), (z_2, \lambda) \in K$.

For any Lebesgue measurable set $M \subset [0, T]$ we denote by $\text{mes}(M)$ the Lebesgue measure of M .

3. LYAPUNOV-SCHMIDT REDUCTION METHOD FOR NONSMOOTH FINITE DIMENSIONAL FUNCTIONS

If the continuously differentiable function $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ vanishes on some set $\mathcal{Z} \subset \mathbb{R}^n$, then sufficient conditions for the existence of zeros near \mathcal{Z} of the perturbed function

$$(3.3) \quad F(z, \varepsilon) = P(z) + \varepsilon Q(z, \varepsilon), \quad z \in \mathbb{R}^n, \quad \varepsilon > 0 \text{ small enough}$$

can be expressed in terms of the restrictions to \mathcal{Z} of the functions $z \mapsto DP(z)$ and $z \mapsto Q(z, 0)$. Roughly speaking, this is what is known in the literature as the Lyapunov–Schmidt reduction method, as it is presented for instance in [8, 4] or [18, §24.8]. In these references it is assumed that Q is a continuously differentiable function. We show in this section that this last assumption can be weakened. The following theorem is the main result of this section and generalizes a theorem of [4].

Theorem 1. *Let $P \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, let $Q \in C^0(\mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ be locally uniformly Lipschitz with respect to its first variable, and let $F : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n$ be given by (3.3). Assume that P satisfies the following hypotheses.*

- (A1) *There exist an invertible $n \times n$ matrix S , an open ball $V \subset \mathbb{R}^k$ with $k \leq n$, and a function $\beta_0 \in C^1(\overline{V}, \mathbb{R}^{n-k})$ such that P vanishes on the set $\mathcal{Z} = \bigcup_{\alpha \in \overline{V}} \left\{ S \begin{pmatrix} \alpha \\ \beta_0(\alpha) \end{pmatrix} \right\}$.*
- (A2) *For any $z \in \mathcal{Z}$ the matrix $DP(z)S$ has in its upper right corner the null $k \times (n-k)$ matrix and in the lower right corner the $(n-k) \times (n-k)$ matrix $\Delta(z)$ with $\det(\Delta(z)) \neq 0$.*

For any $\alpha \in \overline{V}$ we define

$$(3.4) \quad \hat{Q}(\alpha) = \pi Q \left(S \begin{pmatrix} \alpha \\ \beta_0(\alpha) \end{pmatrix}, 0 \right).$$

Then the following statements hold.

- (C1) *For any sequences $(z_m)_{m \geq 1}$ from \mathbb{R}^n and $(\varepsilon_m)_{m \geq 1}$ from $[0, 1]$ such that $z_m \rightarrow z_0 \in \mathcal{Z}$, $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$ and $F(z_m, \varepsilon_m) = 0$ for any $m \geq 1$, we have $\hat{Q}(\pi S^{-1}z_0) = 0$.*
- (C2) *If $\hat{Q} : \overline{V} \rightarrow \mathbb{R}^k$ is such that $\hat{Q}(\alpha) \neq 0$ for all $\alpha \in \partial V$ and $d(\hat{Q}, V) \neq 0$, then there exists $\varepsilon_1 > 0$ sufficiently small such that for each $\varepsilon \in (0, \varepsilon_1]$ there exists at least one $z_\varepsilon \in \mathbb{R}^n$ with $F(z_\varepsilon, \varepsilon) = 0$ and $\rho(z_\varepsilon, \mathcal{Z}) \rightarrow 0$ as $\varepsilon \rightarrow 0$.*

In addition we assume that there exists $\alpha_0 \in V$ such that $\hat{Q}(\alpha_0) = 0$, $\hat{Q}(\alpha) \neq 0$ for all $\alpha \in \overline{V} \setminus \{\alpha_0\}$ and $d(\hat{Q}, V) \neq 0$, and we denote $z_0 = S \begin{pmatrix} \alpha_0 \\ \beta_0(\alpha_0) \end{pmatrix}$. Moreover we also assume:

- (A3) *P is twice differentiable in the points of \mathcal{Z} , and for each $i \in \overline{1, k}$ and $z \in \mathcal{Z}$ the Hessian matrix $HP_i(z)$ is symmetric.*
- (A4) *There exists $\delta_1 > 0$ and $L_{\hat{Q}} > 0$ such that*

$$\|\hat{Q}(\alpha_1) - \hat{Q}(\alpha_2)\| \geq L_{\hat{Q}} \|\alpha_1 - \alpha_2\| \quad \text{for all } \alpha_1, \alpha_2 \in B_{\delta_1}(\alpha_0).$$

- (A5) *For $\delta > 0$ sufficiently small we have that*

$$\begin{aligned} & \|\pi Q(z_1 + \zeta, \varepsilon) - \pi Q(z_1, 0) - \pi Q(z_2 + \zeta, \varepsilon) + \pi Q(z_2, 0)\| \leq \\ & \frac{o(\delta)}{\delta} \|z_1 - z_2\|, \end{aligned}$$

for all $z_1, z_2 \in B_\delta(z_0) \cap \mathcal{Z}$, $\varepsilon \in [0, \delta]$ and $\zeta \in B_\delta(0)$.

Then the following conclusion holds.

(C3) There exists $\delta_2 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1]$ there is exactly one $z_\varepsilon \in B_{\delta_2}(z_0)$ with $F(z_\varepsilon, \varepsilon) = 0$. Moreover $z_\varepsilon \rightarrow z_0$ as $\varepsilon \rightarrow 0$.

We note that a map that satisfy (A4) is usually called *dilating map* (cf. [1]).

For proving Theorem 1 we shall use the following version of the Implicit Function Theorem.

Lemma 1. Let $P \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and let $Q \in C^0(\mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ be locally uniformly Lipschitz with respect to its first variable. Assume that P satisfies the hypotheses (A1) and (A2) of Theorem 1. Then there exist $\delta_0 > 0$, $\varepsilon_0 > 0$ and a function $\beta : \bar{V} \times [0, \varepsilon_0] \rightarrow \mathbb{R}^{n-k}$ such that

(C4) $\pi^\perp F \left(S \begin{pmatrix} \alpha \\ \beta(\alpha, \varepsilon) \end{pmatrix}, \varepsilon \right) = 0$ for all $\alpha \in \bar{V}$ and $\varepsilon \in [0, \varepsilon_0]$.

(C5) $\beta(\alpha, \varepsilon) = \beta_0(\alpha) + \varepsilon \mu(\alpha, \varepsilon)$ where $\mu : \bar{V} \times (0, \varepsilon_0] \rightarrow \mathbb{R}^{n-k}$ is bounded. Moreover for any $\alpha \in \bar{V}$ and $\varepsilon \in [0, \varepsilon_0]$, $\beta(\alpha, \varepsilon)$ is the only zero of $\pi^\perp F \left(S \begin{pmatrix} \alpha \\ \cdot \end{pmatrix}, \varepsilon \right)$ in $B_{\delta_0}(\beta_0(\alpha))$ and β is continuous in $\bar{V} \times [0, \varepsilon_0]$.

In addition if P is twice differentiable in the points of \mathcal{Z} , then

(C6) there exists $L_\mu > 0$ such that $\|\mu(\alpha_1, \varepsilon) - \mu(\alpha_2, \varepsilon)\| \leq L_\mu \|\alpha_1 - \alpha_2\|$ for all $\alpha_1, \alpha_2 \in \bar{V}$ and $\varepsilon \in (0, \varepsilon_0]$.

Proof. (C4) Let $\tilde{F} : \mathbb{R}^k \times \mathbb{R}^{n-k} \times [0, 1] \rightarrow \mathbb{R}^n$ be defined by

$$\tilde{F}(\alpha, \beta, \varepsilon) = F \left(S \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \varepsilon \right),$$

and let \tilde{P} , \tilde{Q} and $\tilde{\Delta}$ be defined in a similar way. Now the assumptions (A1) and (A2) become $\tilde{P}(\alpha, \beta_0(\alpha)) = 0$ and, respectively, the matrix $D\tilde{P}(\alpha, \beta_0(\alpha))$ has in its upper right corner the null $k \times (n-k)$ matrix and in the lower right corner the $(n-k) \times (n-k)$ invertible matrix $\tilde{\Delta}(\alpha, \beta_0(\alpha))$ for any $\alpha \in \bar{V}$. Then

$$\tilde{F}(\alpha, \beta_0(\alpha), 0) = 0 \quad \text{for any } \alpha \in \bar{V},$$

and

$$(3.5) \quad \det \left(D_\beta \left(\pi^\perp \tilde{F} \right) (\alpha, \beta_0(\alpha), 0) \right) = \det \left(\tilde{\Delta}(\alpha, \beta_0(\alpha)) \right) \neq 0 \quad \text{for any } \alpha \in \bar{V}.$$

It follows from (3.5) that there exists a radius $\delta > 0$ such that

$$(3.6) \quad \pi^\perp \tilde{F}(\alpha, \beta, 0) \neq 0 \quad \text{for any } \beta \in \overline{B_\delta(\beta_0(\alpha))} \setminus \{\beta_0(\alpha)\}, \quad \alpha \in \bar{V}.$$

The relations (3.5) and (3.6) give (see [18, Theorem 6.3])

$$d(\pi^\perp \tilde{F}(\alpha, \cdot, 0), B_\delta(\beta_0(\alpha))) = \text{sign} \left(\det(\tilde{\Delta}(\alpha, \beta_0(\alpha))) \right) \neq 0, \quad \alpha \in \bar{V}.$$

Hence, by the continuity of the topological degree with respect to parameters (using the compactness of \overline{V}) there exists $\varepsilon(\delta) > 0$ such that

$$d(\pi^\perp \tilde{F}(\alpha, \cdot, \varepsilon), B_\delta(\beta_0(\alpha))) \neq 0 \quad \text{for any } \varepsilon \in [0, \varepsilon(\delta)], \alpha \in \overline{V}.$$

This assures the existence of $\beta(\alpha, \varepsilon) \in B_\delta(\beta_0(\alpha))$ such that conclusion (C4) holds with $\delta_0 = \delta$ and $\varepsilon_0 = \varepsilon(\delta_0)$.

Without loss of generality we can consider in the sequel that $\varepsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. The value of the radius δ eventually may decrease in a finite number of steps during this proof (consequently, also the value of $\varepsilon(\delta)$). Sometimes we decrease only the value of $\varepsilon(\delta)$, letting δ maintaining its value. Without explicitly mentioning it, finally, in the statement of the lemma, we replace δ_0 by the least value of the radius δ and ε_0 by $\varepsilon(\delta)$.

(C5) Since P and β_0 are C^1 and \overline{V} is bounded, there exists $\eta > 0$ such that the invertible matrix Δ defined by (A2) satisfies $\|\tilde{\Delta}(\alpha, \beta_0(\alpha))\| \geq 2\eta$ for all $\alpha \in \overline{V}$. Using again that P is C^1 and $\tilde{\Delta}(\alpha, \beta_0(\alpha)) = D_\beta(\pi^\perp \tilde{P})(\alpha, \beta_0(\alpha))$, we obtain that the radius $\delta > 0$ found before at (C4) can be decreased, if necessary, in such a way that $\|\tilde{\Delta}(\alpha, \beta_0(\alpha)) - D_\beta(\pi^\perp \tilde{P})(\alpha, \beta)\| \leq \eta$ for all $\beta \in B_\delta(\beta_0(\alpha))$ and $\alpha \in \overline{V}$. Then $\|D_\beta(\pi^\perp \tilde{P})(\alpha, \beta)\| \geq \eta$ for all $\beta \in B_\delta(\beta_0(\alpha))$, $\alpha \in \overline{V}$. Applying the generalized Mean Value Theorem (see [9, Proposition 2.6.5]) to the function $\pi^\perp \tilde{P}(\alpha, \cdot)$, we obtain

$$(3.7) \quad \|\pi^\perp \tilde{P}(\alpha, \beta_1) - \pi^\perp \tilde{P}(\alpha, \beta_2)\| \geq \eta \|\beta_1 - \beta_2\|, \quad \beta_1, \beta_2 \in B_\delta(\beta_0(\alpha)), \alpha \in \overline{V}.$$

We take $M_Q > 0$ such that $\|\tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon)\| \leq M_Q$ for all $\alpha \in \overline{V}$ and $\varepsilon \in [0, \varepsilon_0]$. Using (3.7) we obtain for all $\alpha \in \overline{V}$ and $\varepsilon \in [0, \varepsilon(\delta)]$

$$\begin{aligned} 0 &= \|\pi^\perp \tilde{P}(\alpha, \beta(\alpha, \varepsilon)) - \pi^\perp \tilde{P}(\alpha, \beta_0(\alpha)) + \varepsilon \pi^\perp \tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon)\| \\ &\geq \eta \|\beta(\alpha, \varepsilon) - \beta_0(\alpha)\| - \varepsilon M_Q. \end{aligned}$$

From these last relations, denoting $m = M_Q/\eta$, we deduce that

$$(3.8) \quad \|\mu(\alpha, \varepsilon)\| \leq m \quad \text{for all } \alpha \in \overline{V}, \varepsilon \in (0, \varepsilon(\delta)).$$

We choose $L_Q > 0$ such that

$$(3.9) \quad \|\tilde{Q}(\alpha_2, \beta_2, \varepsilon) - \tilde{Q}(\alpha_1, \beta_1, \varepsilon)\| \leq L_Q (\|\alpha_2 - \alpha_1\| + \|\beta_2 - \beta_1\|),$$

for all $\beta_1, \beta_2 \in B_{\delta_0}(\beta_0(\overline{V}))$, $\alpha_1, \alpha_2 \in \overline{V}$, $\varepsilon \in [0, \varepsilon_0]$. We decrease $\delta > 0$ in such a way that $\eta - \varepsilon L_Q > 0$ for any $\varepsilon \in [0, \varepsilon(\delta)]$.

Let $\alpha \in \overline{V}$, $\varepsilon \in [0, \varepsilon(\delta)]$ and assume that $\beta(\alpha, \varepsilon)$ and β_2 are two zeros of $\pi^\perp F\left(S\left(\begin{smallmatrix} \alpha \\ \cdot \end{smallmatrix}\right), \varepsilon\right)$ in $B_\delta(\beta_0(\alpha))$. Taking into account (3.7) and (3.9), we obtain

$$\begin{aligned} 0 &= \|\pi^\perp \tilde{P}(\alpha, \beta_2) - \pi^\perp \tilde{P}(\alpha, \beta(\alpha, \varepsilon)) + \\ &\quad \varepsilon \pi^\perp \tilde{Q}(\alpha, \beta_2, \varepsilon) - \varepsilon \pi^\perp \tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon)\| \\ &\geq (\eta - \varepsilon L_Q) \|\beta_2 - \beta(\alpha, \varepsilon)\|. \end{aligned}$$

Since $\eta - \varepsilon L_Q > 0$ for any $\varepsilon \in [0, \varepsilon(\delta)]$ we deduce from this last relation that β_2 and $\beta(\alpha, \varepsilon)$ must coincide.

We prove in the sequel the continuity of the function $\beta : \overline{V} \times [0, \varepsilon(\delta)] \rightarrow \mathbb{R}^{n-k}$. Let $(\alpha_1, \varepsilon_1) \in \overline{V} \times [0, \varepsilon(\delta)]$ be fixed and $(\alpha, \varepsilon) \in \overline{V} \times [0, \varepsilon(\delta)]$ be in a small neighborhood of $(\alpha_1, \varepsilon_1)$. Consider $L_P > 0$ such that $\|\tilde{P}(\alpha_1, \beta) - \tilde{P}(\alpha, \beta)\| \leq L_P \|\alpha_1 - \alpha\|$ for all $\alpha_1, \alpha \in \overline{V}$ and $\beta \in B_{\delta_0}(\beta_0(\overline{V}))$. We diminish $\varepsilon(\delta) > 0$, if necessary, and we consider α so close to α_1 that $\beta(\alpha, \varepsilon) \in B_\delta(\beta_0(\alpha_1))$. Then using (3.7) and (3.9) we obtain

$$\begin{aligned} 0 &= \|\pi^\perp \tilde{P}(\alpha_1, \beta(\alpha_1, \varepsilon_1)) - \pi^\perp \tilde{P}(\alpha, \beta(\alpha, \varepsilon)) + \\ &\quad \varepsilon_1 \pi^\perp \tilde{Q}(\alpha_1, \beta(\alpha_1, \varepsilon_1), \varepsilon_1) - \varepsilon \pi^\perp \tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon)\| \\ &\geq \eta \|\beta(\alpha_1, \varepsilon_1) - \beta(\alpha, \varepsilon)\| - L_P \|\alpha_1 - \alpha\| - \\ &\quad \|\varepsilon_1 \pi^\perp \tilde{Q}(\alpha_1, \beta(\alpha_1, \varepsilon_1), \varepsilon_1) - \varepsilon \pi^\perp \tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon)\| \end{aligned}$$

and

$$\begin{aligned} &- \|\varepsilon_1 \pi^\perp \tilde{Q}(\alpha_1, \beta(\alpha_1, \varepsilon_1), \varepsilon_1) - \varepsilon \pi^\perp \tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon)\| \\ &\geq -\varepsilon_1 L_Q \|\alpha_1 - \alpha\| - \varepsilon_1 L_Q \|\beta(\alpha_1, \varepsilon_1) - \beta(\alpha, \varepsilon)\| - \\ &\quad \|\varepsilon_1 \pi^\perp \tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon_1) - \varepsilon \pi^\perp \tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon)\|. \end{aligned}$$

Combining these last two relations we obtain

$$\begin{aligned} (\eta - \varepsilon_1 L_Q) \|\beta(\alpha_1, \varepsilon_1) - \beta(\alpha, \varepsilon)\| &\leq (L_P + \varepsilon_1 L_Q) \|\alpha_1 - \alpha\| + \\ &\quad \|\varepsilon_1 \pi^\perp \tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon_1) - \varepsilon \pi^\perp \tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon)\|, \end{aligned}$$

from where it follows easily that $\beta(\alpha, \varepsilon) \rightarrow \beta(\alpha_1, \varepsilon_1)$ when $(\alpha, \varepsilon) \rightarrow (\alpha_1, \varepsilon_1)$.

(C6) We define $\Phi(\alpha, \xi) = \pi^\perp \tilde{P}(\alpha, \beta_0(\alpha) + \xi)$ for all $\alpha \in \overline{V}$ and $\xi \in \mathbb{R}^{n-k}$. From (3.7) we have that

$$(3.10) \quad \|\Phi(\alpha, \xi_1) - \Phi(\alpha, \xi_2)\| \geq \eta \|\xi_1 - \xi_2\| \text{ for all } \alpha \in \overline{V}, \xi_1, \xi_2 \in B_\delta(0).$$

Since $\tilde{P}(\alpha, \beta_0(\alpha)) = 0$ for all $\alpha \in \overline{V}$, we have that $\Phi(\alpha, \xi) = \pi^\perp \tilde{P}(\alpha, \beta_0(\alpha) + \xi) - \pi^\perp \tilde{P}(\alpha, \beta_0(\alpha))$ and that

$$\begin{aligned} D_\alpha \Phi(\alpha, \xi) &= D_\alpha \left(\pi^\perp \tilde{P} \right) (\alpha, \beta_0(\alpha) + \xi) - D_\alpha \left(\pi^\perp \tilde{P} \right) (\alpha, \beta_0(\alpha)) + \\ &\quad \left[D_\beta \left(\pi^\perp \tilde{P} \right) (\alpha, \beta_0(\alpha) + \xi) - D_\beta \left(\pi^\perp \tilde{P} \right) (\alpha, \beta_0(\alpha)) \right] D\beta_0(\alpha). \end{aligned}$$

From this expression, using that \tilde{P} is twice differentiable in $(\alpha, \beta_0(\alpha))$ and β_0 is C^1 , we obtain for some $L_\Phi > 0$ that the radius δ can be eventually decreased in a such way that

$$\|D_\alpha \Phi(\alpha, \xi)\| \leq L_\Phi \|\xi\| \text{ for all } \alpha \in \overline{V}, \xi \in B_\delta(0).$$

Hence using the Mean Value Inequality we have

$$(3.11) \quad \|\Phi(\alpha_1, \xi) - \Phi(\alpha_2, \xi)\| \leq L_\Phi \|\xi\| \cdot \|\alpha_1 - \alpha_2\| \text{ for all } \alpha_1, \alpha_2 \in \overline{V}, \xi \in B_\delta(0).$$

Now we use (3.10) with $\xi_1 = \varepsilon\mu(\alpha_1, \varepsilon)$, $\xi_2 = \varepsilon\mu(\alpha_2, \varepsilon)$ diminishing $\varepsilon(\delta)$, if necessary, in order that $\xi_1, \xi_2 \in B_\delta(0)$ for all $\alpha_1, \alpha_2 \in \overline{V}$ and $\varepsilon \in (0, \varepsilon(\delta)]$. Using also (C5), (3.8) and (3.11) we obtain

$$(3.12) \quad \begin{aligned} & \|\pi^\perp \tilde{P}(\alpha_1, \beta(\alpha_1, \varepsilon)) - \pi^\perp \tilde{P}(\alpha_2, \beta(\alpha_2, \varepsilon))\| = \|\Phi(\alpha_1, \xi_1) - \Phi(\alpha_2, \xi_2)\| \\ & \geq \eta \|\xi_1 - \xi_2\| - L_\Phi \|\xi_1\| \cdot \|\alpha_1 - \alpha_2\| \\ & \geq \eta \varepsilon \|\mu(\alpha_1, \varepsilon) - \mu(\alpha_2, \varepsilon)\| - L_\Phi m \varepsilon \|\alpha_1 - \alpha_2\|, \end{aligned}$$

for all $\alpha_1, \alpha_2 \in \overline{V}$ and $\varepsilon \in (0, \varepsilon(\delta)]$. Also using (3.9) we have

$$(3.13) \quad \begin{aligned} & \|\pi^\perp \tilde{Q}(\alpha_1, \beta(\alpha_1, \varepsilon), \varepsilon) - \pi^\perp \tilde{Q}(\alpha_2, \beta(\alpha_2, \varepsilon), \varepsilon)\| \leq \\ & \leq \varepsilon L_Q \|\mu(\alpha_1, \varepsilon) - \mu(\alpha_2, \varepsilon)\| + L_Q(1 + L_{\beta_0}) \|\alpha_1 - \alpha_2\|, \end{aligned}$$

for all $\alpha_1, \alpha_2 \in \overline{V}$ and $\varepsilon \in (0, \varepsilon(\delta)]$, where L_{β_0} is the Lipschitz constant of β_0 in \overline{V} . By definition of $\beta(\alpha, \varepsilon)$ we have $\pi^\perp \tilde{P}(\alpha_i, \beta(\alpha_i, \varepsilon)) + \varepsilon \pi^\perp \tilde{Q}(\alpha_i, \beta(\alpha_i, \varepsilon), \varepsilon) = 0$ for $i \in \overline{1, 2}$. Using (3.12) and (3.13) we obtain

$$0 \geq \varepsilon[\eta - \varepsilon L_Q] \cdot \|\mu(\alpha_1, \varepsilon) - \mu(\alpha_2, \varepsilon)\| - \varepsilon[L_\Phi m + L_Q(1 + L_{\beta_0})] \cdot \|\alpha_1 - \alpha_2\|,$$

for all $\alpha_1, \alpha_2 \in \overline{V}$ and $\varepsilon \in (0, \varepsilon(\delta)]$. Therefore $\mu : \overline{V} \times (0, \varepsilon(\delta)] \rightarrow \mathbb{R}^{n-k}$ satisfies (C6) with $L_\mu = [L_\Phi m + L_Q(1 + L_{\beta_0})]/[\eta - \varepsilon(\delta)L_Q]$. Hence all the conclusions hold with $\delta_0 = \delta$ and $\varepsilon_0 = \varepsilon(\delta)$. \square

We remark that (C4) and the uniqueness part of (C5) can be obtained by means of the Lipschitz generalization of the Inverse Function Theorem (see e.g. [16, Theorem 5.3.8]), but we provide a different proof because the inequalities (3.7) and (3.8) are used for proving the rest of (C5) and (C6).

Proof of Theorem 1. Let $\delta_0, \varepsilon_0, \beta(\alpha, \varepsilon)$ and $\mu(\alpha, \varepsilon)$ be as in Lemma 1. We consider the notations \tilde{F}, \tilde{P} and \tilde{Q} like in the proof of Lemma 1.

(C1) Let the sequences $(z_m)_{m \geq 1}$ from \mathbb{R}^n and $(\varepsilon_m)_{m \geq 1}$ from $[0, 1]$ be such that $z_m \rightarrow z_0 \in \mathcal{Z}$, $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$ and $F(z_m, \varepsilon_m) = 0$ for any $m \geq 1$. We define $\alpha_0 \in \mathbb{R}^k$, the sequences $(\alpha_m)_{m \geq 1}$ from \mathbb{R}^k and $(\beta_m)_{m \geq 1}$ from \mathbb{R}^{n-k} by $z_0 = S \begin{pmatrix} \alpha_0 \\ \beta_0(\alpha_0) \end{pmatrix}$ and $z_m = S \begin{pmatrix} \alpha_m \\ \beta_m \end{pmatrix}$. Then we have that $\alpha_0 = \lim_{m \rightarrow \infty} \alpha_m$, $\beta_0(\alpha_0) = \lim_{m \rightarrow \infty} \beta_m$ and there exists $m_0 \in \mathbb{N}$ such that $\beta_m \in B_{\delta_0}(\beta_0(\alpha_m))$ and $\varepsilon_m \in [0, \varepsilon_0]$ for all $m \geq m_0$. Therefore, since $F(z_m, \varepsilon_m) = 0$, Lemma 1 implies $\beta_m = \beta(\alpha_m, \varepsilon_m)$ for any $m \geq m_0$. Since $\pi \tilde{P}(\alpha_m, \beta_0(\alpha_m)) = 0$ and $D_\beta(\pi \tilde{P})(\alpha_m, \beta_0(\alpha_m)) = 0$, we obtain that $\lim_{m \rightarrow \infty} \frac{1}{\varepsilon_m} \pi \tilde{P}(\alpha_m, \beta(\alpha_m, \varepsilon_m)) = 0$. Hence

$$\begin{aligned} 0 &= \lim_{m \rightarrow \infty} \frac{1}{\varepsilon_m} \pi \tilde{F}(\alpha_m, \beta(\alpha_m, \varepsilon_m), \varepsilon_m) \\ &= \lim_{m \rightarrow \infty} \left[\frac{1}{\varepsilon_m} \pi \tilde{P}(\alpha_m, \beta(\alpha_m, \varepsilon_m)) + \pi \tilde{Q}(\alpha_m, \beta(\alpha_m, \varepsilon_m), \varepsilon_m) \right] = \hat{Q}(\alpha_0) \end{aligned}$$

from where (C1) follows.

(C2) Using (C4) of Lemma 1, we note that it is enough to prove the existence of at least one zero in V of the function $\alpha \mapsto \pi\tilde{F}(\alpha, \beta(\alpha, \varepsilon), \varepsilon)$ for each $\varepsilon \in (0, \varepsilon_1]$ where ε_1 with $0 < \varepsilon_1 \leq \varepsilon_0$ has to be found. This will follow from the claim that the Brouwer topological degree $d\left(\frac{1}{\varepsilon}\pi\tilde{F}(\cdot, \beta(\cdot, \varepsilon), \varepsilon), V\right) \neq 0$ for $\varepsilon \in (0, \varepsilon_1]$. Now we prove this claim. Since $\beta(\alpha, \varepsilon) = \beta_0(\alpha) + \varepsilon\mu(\alpha, \varepsilon)$ with $\mu : \overline{V} \times (0, \varepsilon_0] \rightarrow \mathbb{R}^{n-k}$ a bounded function, $\pi\tilde{P}(\alpha, \beta_0(\alpha)) = 0$ and $D_\beta(\pi\tilde{P})(\alpha, \beta_0(\alpha)) = 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \pi\tilde{P}(\alpha, \beta(\alpha, \varepsilon)) = 0.$$

Therefore

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \pi\tilde{F}(\alpha, \beta(\alpha, \varepsilon), \varepsilon) = \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon} \pi\tilde{P}(\alpha, \beta(\alpha, \varepsilon)) + \pi\tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon) \right] = \hat{Q}(\alpha).$$

Using the continuity of the Brouwer degree with respect to the parameter ε , and taking into account that, by hypothesis, $d(\hat{Q}, V) \neq 0$, for each $\varepsilon \in (0, \varepsilon_1]$ there exists $\varepsilon_1 > 0$ sufficiently small such that

$$d\left(\frac{1}{\varepsilon}\pi\tilde{F}(\cdot, \beta(\cdot, \varepsilon), \varepsilon), V\right) = d(\hat{Q}, V) \neq 0.$$

Hence the claim is proved. Then for each $\varepsilon \in (0, \varepsilon_1]$ there exists $\alpha_\varepsilon \in V$ such that $\pi\tilde{F}(\alpha_\varepsilon, \beta(\alpha_\varepsilon, \varepsilon), \varepsilon) = 0$ and, moreover, using also (C4) of Lemma 1, we have that $\tilde{F}(\alpha_\varepsilon, \beta(\alpha_\varepsilon, \varepsilon), \varepsilon) = 0$. Denoting $z_\varepsilon = S\left(\begin{smallmatrix} \alpha_\varepsilon \\ \beta(\alpha_\varepsilon, \varepsilon) \end{smallmatrix}\right)$ we have that $F(z_\varepsilon, \varepsilon) = 0$. From the definitions of z_ε and \mathcal{Z} , and the continuity of β , it follows easily that $\rho(z_\varepsilon, \mathcal{Z}) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

(C3) Since $\alpha_0 \in V$ is an isolated zero of \hat{Q} , applying the topological degree arguments like in (C2) for V that shrinks to $\{\alpha_0\}$, we obtain the existence of α_ε such that $\alpha_\varepsilon \rightarrow \alpha_0$ as $\varepsilon \rightarrow 0$, and $\pi\tilde{F}(\alpha_\varepsilon, \beta(\alpha_\varepsilon, \varepsilon), \varepsilon) = 0$ for any $\varepsilon \in (0, \varepsilon_1]$. Hence $z_\varepsilon = S\left(\begin{smallmatrix} \alpha_\varepsilon \\ \beta(\alpha_\varepsilon, \varepsilon) \end{smallmatrix}\right)$ and $z_0 = S\left(\begin{smallmatrix} \alpha_0 \\ \beta_0(\alpha_0) \end{smallmatrix}\right) \in \mathcal{Z}$ are such that $F(z_\varepsilon, \varepsilon) = 0$ and $z_\varepsilon \rightarrow z_0$ as $\varepsilon \rightarrow 0$.

In order to prove that z_ε is the unique zero of $F(\cdot, \varepsilon)$ in a neighborhood of z_0 , we define

$$r_1(\alpha, \varepsilon) = \frac{1}{\varepsilon} \pi\tilde{P}(\alpha, \beta(\alpha, \varepsilon)), \quad r_2(\alpha, \varepsilon) = \pi\tilde{Q}(\alpha, \beta(\alpha, \varepsilon), \varepsilon) - \pi\tilde{Q}(\alpha, \beta_0(\alpha), 0),$$

for all $\alpha \in \overline{V}$ and $\varepsilon \in (0, \varepsilon_1]$, and we study the Lipschitz properties with respect to α of these two functions.

Since $\tilde{P}(\alpha, \beta_0(\alpha)) = 0$ for all $\alpha \in \overline{V}$, by taking the derivative with respect to α we obtain

$$(3.14) \quad D_\alpha(\pi\tilde{P})(\alpha, \beta_0(\alpha)) + D_\beta(\pi\tilde{P})(\alpha, \beta_0(\alpha))D\beta_0(\alpha) = 0 \quad \text{for all } \alpha \in \overline{V}.$$

Assumption (A2) assures that $D_\beta (\pi \tilde{P}) (\alpha, \beta_0(\alpha)) = 0$ for all $\alpha \in \overline{V}$. Taking the derivative with respect to α , we have

$$(3.15) \quad D_{\beta\alpha} (\pi \tilde{P}) (\alpha, \beta_0(\alpha)) + D_{\beta\beta} (\pi \tilde{P}) (\alpha, \beta_0(\alpha)) D\beta_0(\alpha) = 0 \quad \text{for any } \alpha \in \overline{V}.$$

For any $\alpha \in \overline{V}$ and $\xi \in \mathbb{R}^{n-k}$ we define $\Phi(\alpha, \xi) = \pi \tilde{P}(\alpha, \beta_0(\alpha) + \xi)$. Taking into account the relations (3.14) and (3.15) and that, by hypothesis (A3) we have that $D_{\beta\alpha} (\pi \tilde{P}) (\alpha, \beta_0(\alpha)) = D_{\alpha\beta} (\pi \tilde{P}) (\alpha, \beta_0(\alpha))$, we obtain

$$\begin{aligned} D_\alpha \Phi(\alpha, \xi) &= D_\alpha (\pi \tilde{P}) (\alpha, \beta_0(\alpha) + \xi) + D_\beta (\pi \tilde{P}) (\alpha, \beta_0(\alpha) + \xi) D\beta_0(\alpha) - \\ &\quad D_\alpha (\pi \tilde{P}) (\alpha, \beta_0(\alpha)) - D_\beta (\pi \tilde{P}) (\alpha, \beta_0(\alpha)) D\beta_0(\alpha) - \\ &\quad D_{\alpha\beta} (\pi \tilde{P}) (\alpha, \beta_0(\alpha)) \xi - D_{\beta\beta} (\pi \tilde{P}) (\alpha, \beta_0(\alpha)) D\beta_0(\alpha) \xi. \end{aligned}$$

From this last equality, using that $D_\alpha (\pi \tilde{P})$ and, respectively, $D_\beta (\pi \tilde{P})$ are differentiable at $(\alpha, \beta_0(\alpha))$, we deduce that $D_\alpha \Phi(\alpha, \xi) = o(\xi)$ for all $\alpha \in \overline{V}$ and $\xi \in \mathbb{R}^{n-k}$ with $\|\xi\|$ sufficiently small. Hence the mean value inequality assures that

$$\|\Phi(\alpha_1, \xi) - \Phi(\alpha_2, \xi)\| \leq o(\xi) \|\alpha_1 - \alpha_2\| \quad \text{for all } \alpha_1, \alpha_2 \in \overline{V}.$$

In the last inequality we replace $\xi = \varepsilon \mu(\alpha_1, \varepsilon)$ (where μ is given by Lemma 1). We use that $D_\xi \Phi(\alpha, 0) = D_\beta \pi \tilde{P}(\alpha, \beta_0(\alpha)) = 0$ for any $\alpha \in \overline{V}$, and that μ is Lipschitz with respect to $\alpha \in \overline{V}$. Then we obtain, considering that ε_1 is small enough, for all $\varepsilon \in (0, \varepsilon_1]$

$$\|\Phi(\alpha_1, \varepsilon \mu(\alpha_1, \varepsilon)) - \Phi(\alpha_2, \varepsilon \mu(\alpha_2, \varepsilon))\| \leq o(\varepsilon) \|\alpha_1 - \alpha_2\| \quad \text{for all } \alpha_1, \alpha_2 \in V.$$

Now coming back to our notations and recalling that $\beta(\alpha, \varepsilon) = \beta_0(\alpha) + \varepsilon \mu(\alpha, \varepsilon)$, we obtain for $\varepsilon \in (0, \varepsilon_1]$

$$(3.16) \quad \|r_1(\alpha_1, \varepsilon) - r_1(\alpha_2, \varepsilon)\| \leq \frac{o(\varepsilon)}{\varepsilon} \|\alpha_1 - \alpha_2\| \quad \text{for all } \alpha_1, \alpha_2 \in \overline{V}.$$

We will prove that a similar relation holds for the function r_2 . First we note that the hypothesis (A5) and the fact that Q is locally uniformly Lipschitz with respect to the first variable imply that

$$(3.17) \quad \|\pi Q(z_1 + \zeta_1, \varepsilon) - \pi Q(z_1, 0) - \pi Q(z_2 + \zeta_2, \varepsilon) + \pi Q(z_2, 0)\| \leq \frac{o(\delta)}{\delta} \|z_1 - z_2\| + L_Q \|\zeta_1 - \zeta_2\|,$$

for all $z_1, z_2 \in B_\delta(z_0) \cap \mathcal{Z}$, $\varepsilon \in [0, \delta]$ and $\zeta_1, \zeta_2 \in B_\delta(0)$. We diminish $\delta_1 > 0$ given in (A4) and $\varepsilon_1 > 0$ in such a way that $\delta_1 \leq \delta$, $\varepsilon_1 \leq \delta$, $S \begin{pmatrix} \alpha \\ \beta_0(\alpha) \end{pmatrix} \in B_\delta(z_0)$ and $S \begin{pmatrix} 0_{k \times 1} \\ \varepsilon \mu(\alpha, \varepsilon) \end{pmatrix} \in B_\delta(0)$ for any $\alpha \in B_{\delta_1}(\alpha_0)$, $\varepsilon \in (0, \varepsilon_1]$. Replacing

$z_i = S \begin{pmatrix} \alpha_i \\ \beta_0(\alpha_i) \end{pmatrix}$, $\zeta_i = S \begin{pmatrix} 0_{k \times 1} \\ \varepsilon \mu(\alpha_i, \varepsilon) \end{pmatrix}$, $i \in \overline{1, 2}$ in (3.17) we obtain that

$$\begin{aligned} & \|r_2(\alpha_1, \varepsilon) - r_2(\alpha_2, \varepsilon)\| \leq \\ & \leq \frac{o(\delta)}{\delta} (\|\alpha_1 - \alpha_2\| + \|\beta_0(\alpha_1) - \beta_0(\alpha_2)\|) + \varepsilon L_Q \|\mu(\alpha_1, \varepsilon) - \mu(\alpha_2, \varepsilon)\|, \end{aligned}$$

for all $\alpha_1, \alpha_2 \in B_{\delta_1}(\alpha_0)$ and $\varepsilon \in (0, \varepsilon_1]$. By hypothesis, β_0 is C^1 in \overline{V} and, by Lemma 1 (conclusion (C6)), $(\alpha, \varepsilon) \mapsto \mu(\alpha, \varepsilon)$ is Lipschitz with respect to $\alpha \in \overline{V}$ (with a Lipschitz constant that does not depend on ε). Hence for $\delta_1, \varepsilon_1 \leq \delta$ small enough,

$$(3.18) \quad \|r_2(\alpha_1, \varepsilon) - r_2(\alpha_2, \varepsilon)\| \leq \frac{o(\delta)}{\delta} \|\alpha_1 - \alpha_2\|, \quad \alpha_1, \alpha_2 \in B_{\delta_1}(\alpha_0), \varepsilon \in (0, \varepsilon_1].$$

Therefore we have proved that r_1 and r_2 satisfy the Lipschitz conditions (3.16) and, respectively, (3.18). In what follows we define some constant $\delta_2 > 0$, and after we prove that it is the one that satisfies the requirements of (C3).

We diminish $\delta_1 > 0$ in such a way that there exists $\delta_3 > 0$ such that $\delta_3 \leq \delta_0$ and $B_{\delta_3}(\beta_0(\alpha_0)) \subset \bigcap_{\alpha \in B_{\delta_1}(\alpha_0)} B_{\delta_0}(\beta_0(\alpha))$. We choose $\delta_2 > 0$ so small that $S^{-1}(B_{\delta_2}(z_0)) \subset B_{\delta_1}(\alpha_0) \times B_{\delta_3}(\beta_0(\alpha_0))$. We diminish $\varepsilon_1 > 0$, if necessary, such that $z_\varepsilon \in B_{\delta_2}(z_0)$ for any $\varepsilon \in (0, \varepsilon_1]$. For any $\varepsilon \in (0, \varepsilon_1]$ we claim that z_ε is the only zero of $F(\cdot, \varepsilon)$ in $B_{\delta_2}(z_0)$. Assume by contradiction that there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that z_{ε_2} and z_2 are two different zeros of $F(\cdot, \varepsilon_2)$ in $B_{\delta_2}(z_0)$. Denoting $\alpha_2 = \pi S^{-1} z_2$ and $\beta_2 = \pi^\perp S^{-1} z_2$ we have that $\beta_2 \in B_{\delta_0}(\beta_0(\alpha_2))$. By (C5) of Lemma 1, since β_2 is a zero of $\pi^\perp F \left(S \begin{pmatrix} \alpha_2 \\ \cdot \end{pmatrix}, \varepsilon_2 \right)$ (using the notations introduced before, $\pi^\perp \tilde{F}(\alpha_2, \cdot, \varepsilon_2)$), we must have that $\beta_2 = \beta(\alpha_2, \varepsilon_2)$. Therefore α_{ε_2} and α_2 are two different zeros of $\pi \tilde{F}(\cdot, \beta(\cdot, \varepsilon_2), \varepsilon_2)$ in $B_{\delta_1}(\alpha_0)$. We have the identity

$$\frac{1}{\varepsilon} \pi \tilde{F}(\alpha, \beta(\alpha, \varepsilon), \varepsilon) = \hat{Q}(\alpha) + r_1(\alpha, \varepsilon) + r_2(\alpha, \varepsilon) \text{ for all } \alpha \in \overline{V}, \varepsilon \in (0, \varepsilon_1].$$

We denote $r(\alpha, \varepsilon) = r_1(\alpha, \varepsilon) + r_2(\alpha, \varepsilon)$. Then assumption (A4), properties (3.16) and (3.18) give

$$0 = \|\hat{Q}(\alpha_{\varepsilon_2}) - \hat{Q}(\alpha_2) + r(\alpha_{\varepsilon_2}, \varepsilon_2) - r(\alpha_2, \varepsilon_2)\| \geq (L_{\hat{Q}} - o(\varepsilon_2)/\varepsilon_2 - o(\delta)/\delta) \|\alpha_{\varepsilon_2} - \alpha_2\|.$$

Since $\varepsilon_1 > 0$ and $\delta > 0$ are sufficiently small and $0 < \varepsilon_2 \leq \varepsilon_1$, the constant $(L_{\hat{Q}} - o(\varepsilon_2)/\varepsilon_2 - o(\delta)/\delta)$ must be positive and, consequently, α_{ε_2} and α_2 must coincide. Hence also z_{ε_2} and z_2 must coincide and we conclude the proof. \square

4. A GENERALIZATION OF MALKIN'S RESULT ON THE EXISTENCE OF T -PERIODIC SOLUTIONS FOR T -PERIODICALLY PERTURBED DIFFERENTIAL EQUATIONS WHEN THE PERTURBATION IS NONSMOOTH

In this section we consider the problem of existence and uniqueness of T -periodic solutions for the T -periodic differential system

$$(4.19) \quad \dot{x} = f(t, x) + \varepsilon g(t, x, \varepsilon),$$

where $f \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C^0(\mathbb{R} \times \mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ are T -periodic in the first variable and g is locally uniformly Lipschitz with respect to its second variable. For $z \in \mathbb{R}^n$ we denote by $x(\cdot, z, \varepsilon)$ the solution of (4.19) such that $x(0, z, \varepsilon) = z$. We consider the situation when the unperturbed system

$$(4.20) \quad \dot{x} = f(t, x),$$

has a non-degenerate (in a sense that will be precised below) family of T -periodic solutions. The main tool for the proof of our main result is Theorem 1. We will show that the assumptions of Theorem 1 can be expressed in terms of the function g and of the solutions of the linear differential system

$$(4.21) \quad \dot{y} = D_x f(t, x(t, z, 0))y.$$

Indeed we have the following theorem, which generalizes a related result by Roseau and improves it with the uniqueness of the periodic solution. The above mentioned result by Roseau is proved in a shorter way in [4]. Here we will use the same main ideas from [4] to prove the next result.

Theorem 2. *Assume that $f \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C^0(\mathbb{R} \times \mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ are T -periodic in the first variable, and that g is locally uniformly Lipschitz with respect to the second variable. Suppose that the unperturbed system (4.20) satisfies the following conditions.*

- (A6) *There exist an invertible $n \times n$ real matrix S , an open ball $V \subset \mathbb{R}^k$ with $k \leq n$, and a C^1 function $\beta_0 : \overline{V} \rightarrow \mathbb{R}^{n-k}$ such that any point of the set $\mathcal{Z} = \bigcup_{\alpha \in \overline{V}} \left\{ S \begin{pmatrix} \alpha \\ \beta_0(\alpha) \end{pmatrix} \right\}$ is the initial condition of a T -periodic solution of (4.20).*
- (A7) *For each $z \in \mathcal{Z}$ there exists a fundamental matrix solution $Y(\cdot, z)$ of (4.21) such that $Y(0, z)$ is C^1 with respect to z and $Y^{-1}(0, z) - Y^{-1}(T, z)$ has in the upper right corner the null $k \times (n - k)$ matrix, while in the lower right corner has the $(n - k) \times (n - k)$ matrix $\Delta(z)$ with $\det(\Delta(z)) \neq 0$.*

We define the function $G : \overline{V} \rightarrow \mathbb{R}^k$ by

$$G(\alpha) = \pi \int_0^T Y^{-1} \left(t, S \begin{pmatrix} \alpha \\ \beta_0(\alpha) \end{pmatrix} \right) g \left(t, x \left(t, S \begin{pmatrix} \alpha \\ \beta_0(\alpha) \end{pmatrix}, 0 \right), 0 \right) dt.$$

Then the following statements hold.

(C7) For any sequences $(\varphi_m)_{m \geq 1}$ from $C^0(\mathbb{R}, \mathbb{R}^n)$ and $(\varepsilon_m)_{m \geq 1}$ from $[0, 1]$ such that $\varphi_m(0) \rightarrow z_0 \in \mathcal{Z}$, $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$ and φ_m is a T -periodic solution of (4.19) with $\varepsilon = \varepsilon_m$ for any $m \geq 1$, we have that $G(\pi S^{-1}z_0) = 0$.

(C8) If $G(\alpha) \neq 0$ for any $\alpha \in \partial V$ and $d(G, V) \neq 0$, then there exists $\varepsilon_1 > 0$ sufficiently small such that for each $\varepsilon \in (0, \varepsilon_1]$ there is at least one T -periodic solution φ_ε of system (4.19) such that $\rho(\varphi_\varepsilon(0), \mathcal{Z}) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

In addition we assume that there exists $\alpha_0 \in V$ such that $G(\alpha_0) = 0$, $G(\alpha) \neq 0$ for all $\alpha \in \overline{V} \setminus \{\alpha_0\}$ and $d(G, V) \neq 0$, and we denote $z_0 = S \begin{pmatrix} \alpha_0 \\ \beta_0(\alpha_0) \end{pmatrix}$. Moreover we also assume:

(A8) There exists $\delta_1 > 0$ and $L_G > 0$ such that

$$\|G(\alpha_1) - G(\alpha_2)\| \geq L_G \|\alpha_1 - \alpha_2\|, \text{ for all } \alpha_1, \alpha_2 \in B_{\delta_1}(\alpha_0),$$

(A9) For $\delta > 0$ sufficiently small there exists $M_\delta \subset [0, T]$ Lebesgue measurable with $\text{mes}(M_\delta) = o(\delta)/\delta$ such that

$$\|g(t, z_1 + \zeta, \varepsilon) - g(t, z_1, 0) - g(t, z_2 + \zeta, \varepsilon) + g(t, z_2, 0)\| \leq o(\delta)/\delta \|z_1 - z_2\|,$$

for all $t \in [0, T] \setminus M_\delta$ and for all $z_1, z_2 \in B_\delta(z_0)$, $\varepsilon \in [0, \delta]$ and $\zeta \in B_\delta(0)$.

Then the following conclusion holds.

(C9) There exists $\delta_2 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, φ_ε is the only T -periodic solution of (4.19) with initial condition in $B_{\delta_2}(z_0)$. Moreover $\varphi_\varepsilon(0) \rightarrow z_0$ as $\varepsilon \rightarrow 0$.

To prove the theorem we need three preliminary lemmas that are interesting by themselves. For example, in Lemma 3 we prove the existence of the derivative (in $\varepsilon = 0$) with respect to some parameter denoted ε of the solution of some initial value problem without assuming that the system is C^1 . We also study the properties of this derivative.

Lemma 2. Let $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and K_1, K_2 be compact subsets of \mathbb{R}^n . Then the following inequality holds for all $x_1^0, x_2^0 \in K_1$, $y_1, y_2 \in K_2$ and $\varepsilon \in [0, 1]$.

$$(4.22) \quad \|f(x_1^0 + \varepsilon y_1) - f(x_1^0) - f(x_2^0 + \varepsilon y_2) + f(x_2^0)\| \leq O(\varepsilon) \|x_1^0 - x_2^0\| + O(\varepsilon) \|y_1 - y_2\|.$$

In addition for $m > 0$ sufficiently small and $u_1, u_2, v_1, v_2 \in B_m(0) \subset \mathbb{R}^n$ we have

$$(4.23) \quad \begin{aligned} & \|f(x_1^0 + v_1 + \varepsilon y_1^0 + \varepsilon u_1) - f(x_1^0 + v_1) - \varepsilon f'(x_1^0)y_1^0 - \\ & f(x_2^0 + v_2 + \varepsilon y_2^0 + \varepsilon u_2) + f(x_2^0 + v_2) + \varepsilon f'(x_2^0)y_2^0\| \leq \\ & [o(\varepsilon) + \varepsilon O(m)] \|x_1^0 - x_2^0\| + O(\varepsilon) \|v_1 - v_2\| + \\ & [o(\varepsilon) + \varepsilon O(m)] \|y_1^0 - y_2^0\| + O(\varepsilon) \|u_1 - u_2\|. \end{aligned}$$

Proof. We define $\Phi(x^0, y, \varepsilon) = f(x^0 + \varepsilon y) - f(x^0)$ for all $x^0 \in \overline{\text{co}}K_1$, $y \in \overline{\text{co}}K_2$ and $\varepsilon \in [0, 1]$. Relation (4.22) follows from the mean value inequality applied to

Φ_i with $i \in \overline{1, n}$ and the following estimations.

$$\begin{aligned}\frac{\partial \Phi_i}{\partial x^0}(x^0, y, \varepsilon) &= (f_i)'(x^0 + \varepsilon y) - (f_i)'(x^0) = O(\varepsilon) \quad \text{and} \\ \frac{\partial \Phi_i}{\partial y}(x^0, y, \varepsilon) &= \varepsilon (f_i)'(x^0 + \varepsilon y) = O(\varepsilon).\end{aligned}$$

In order to prove relation (4.23) we define

$$\Phi(x^0, v, y^0, u, \varepsilon) = f(x^0 + v + \varepsilon y^0 + \varepsilon u) - f(x^0 + v) - \varepsilon f'(x^0) y^0,$$

for all $x^0 \in \overline{\text{co}}K_1$, $y^0 \in \overline{\text{co}}K_2$, $u, v \in B_m(0)$ and $\varepsilon \in [0, 1]$. We apply again the mean value inequality to the components Φ_i , $i \in \overline{1, n}$, using the following estimations.

$$\begin{aligned}\frac{\partial \Phi_i}{\partial x^0}(x^0, v, y^0, u, \varepsilon) &= (f_i)'(x^0 + v + \varepsilon y^0 + \varepsilon u) - (f_i)'(x^0 + v) - \varepsilon (f_i)''(x^0) y^0 \\ &= o(\varepsilon) + \varepsilon (f_i)''(x^0 + v) u \\ &\quad + \varepsilon [(f_i)''(x^0 + v) - (f_i)''(x^0)] y^0 \\ &= o(\varepsilon) + \varepsilon O(m) + \varepsilon o(m)/m = o(\varepsilon) + \varepsilon O(m), \\ \frac{\partial \Phi}{\partial v}(x^0, v, y^0, u, \varepsilon) &= (f_i)'(x^0 + v + \varepsilon y^0 + \varepsilon u) - (f_i)'(x^0 + v) = O(\varepsilon), \\ \frac{\partial \Phi_i}{\partial y^0}(x^0, v, y^0, u, \varepsilon) &= \varepsilon (f_i)'(x^0 + v + \varepsilon y^0 + \varepsilon u) - \varepsilon (f_i)'(x^0) \\ &= \varepsilon (f_i)'(x^0 + v + \varepsilon y^0 + \varepsilon u) - \varepsilon (f_i)'(x^0 + v) \\ &\quad + \varepsilon (f_i)'(x^0 + v) - \varepsilon (f_i)'(x^0) \\ &= o(\varepsilon) + \varepsilon O(m), \\ \frac{\partial \Phi}{\partial u}(x^0, v, y^0, u, \varepsilon) &= \varepsilon (f_i)'(x^0 + v + \varepsilon y^0 + \varepsilon u) = O(\varepsilon).\end{aligned}$$

□

Lemma 3. *We consider $f \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C^0(\mathbb{R} \times \mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ a locally uniformly Lipschitz function with respect to the second variable. For $z \in \mathbb{R}^n$ and $\varepsilon \in [0, 1]$, we denote by $x(\cdot, z, \varepsilon)$ the unique solution of*

$$\dot{x} = f(t, x) + \varepsilon g(t, x, \varepsilon), \quad x(0) = z,$$

and by $y(t, z, \varepsilon) = [x(t, z, \varepsilon) - x(t, z, 0)] / \varepsilon$ (here $\varepsilon \neq 0$). We assume that for a given $T > 0$ there exist a compact set $K \subset \mathbb{R}^n$ with nonempty interior and $\delta > 0$ such that $x(t, z, \varepsilon)$ is well-defined for all $t \in [0, T]$, $z \in K$ and $\varepsilon \in [0, \delta]$. Then the following statements hold.

(C10) *There exists $y(t, z, 0) = \lim_{\varepsilon \rightarrow 0} y(t, z, \varepsilon)$ being the solution of the initial value problem*

$$\dot{y}(t) = D_x f(t, x(t, z, 0)) y + g(t, x(t, z, 0), 0), \quad y(0) = 0.$$

The above limit holds uniformly with respect to $(t, z) \in [0, T] \times K$.

(C11) *The functions $x, y : [0, T] \times K \times [0, \delta] \rightarrow \mathbb{R}^n$ are continuous and uniformly Lipschitz with respect to their second variable.*

(C12) *In addition if there exists $z_0 \in \text{int}(K)$ such that assumption (A9) of Theorem 2 holds with the same small $\delta > 0$ as above, then*

$$\|y(t, z_1 + \zeta, \varepsilon) - y(t, z_1, 0) - y(t, z_2 + \zeta, \varepsilon) + y(t, z_2, 0)\| \leq o(\delta)/\delta \|z_1 - z_2\|,$$

for all $t \in [0, T]$, $z_1, z_2 \in B_\delta(z_0)$, $\varepsilon \in [0, \delta]$ and $\zeta \in B_\delta(0)$.

Proof. (C10) We define $\tilde{f}(t, z, \varepsilon) = \frac{f(t, x(t, z, \varepsilon)) - f(t, x(t, z, 0))}{x(t, z, \varepsilon) - x(t, z, 0)}$ for $\varepsilon \neq 0$ and $\tilde{f}(t, z, 0) = D_x f(t, x(t, z, 0))$. In this way we obtain the continuous function $\tilde{f} : [0, T] \times K \times [0, \delta] \rightarrow \mathbb{R}^n$. For $\varepsilon \neq 0$, using the definitions of $x(t, z, \varepsilon)$ and $y(t, z, \varepsilon)$ we deduce immediately that $y(0, z, \varepsilon) = 0$ and also that

$$(4.24) \quad \dot{y}(t, z, \varepsilon) = \tilde{f}(t, x(t, z, \varepsilon))y(t, z, \varepsilon) + g(t, x(t, z, \varepsilon), \varepsilon).$$

Passing to the limit as $\varepsilon \rightarrow 0$, we obtain that $y(\cdot, z, 0)$ is the solution of the given initial value problem. Hence (4.24) holds also for $\varepsilon = 0$. Since the right hand side of (4.24) is given by a continuous function, we have that the limit $y(t, z, 0) = \lim_{\varepsilon \rightarrow 0} y(t, z, \varepsilon)$ holds uniformly with respect to $(t, z) \in [0, T] \times K$.

(C11) The facts that the functions $x, y : [0, T] \times K \times [0, \delta] \rightarrow \mathbb{R}^n$ are continuous, and that x is Lipschitz with respect to its second variable can be obtained as a corollary of the general theorem on the dependence of the solutions of an ordinary differential equation on the parameters (see [2, Lemma 8.2]).

It remains to prove that $y : [0, T] \times K \times [0, \delta] \rightarrow \mathbb{R}^n$ is uniformly Lipschitz with respect to its second variable.

There exist compact subsets K_1 and K_2 of \mathbb{R}^n such that $x(t, z, \varepsilon) \in K_1$ and $y(t, z, \varepsilon) \in K_2$ for all $(t, z, \varepsilon) \in [0, T] \times K \times [0, \delta]$.

Moreover the representation $x(s, z, \varepsilon) = x(s, z, 0) + \varepsilon y(s, z, \varepsilon)$ allows to use Lemma 2, relation (4.22) with $x_1^0 = x(s, z_1, 0)$, $x_2^0 = x(s, z_2, 0)$, $y_1 = y(s, z_1, \varepsilon)$, $y_2 = y(s, z_2, \varepsilon)$ in order to obtain

$$\begin{aligned} & \|f(t, x(t, z_1, \varepsilon)) - f(t, x(t, z_1, 0)) - f(t, x(t, z_2, \varepsilon)) + f(t, x(t, z_2, 0))\| \leq \\ & O(\varepsilon) \|x(t, z_1, 0) - x(t, z_2, 0)\| + O(\varepsilon) \|y(t, z_1, \varepsilon) - y(t, z_2, \varepsilon)\|, \end{aligned}$$

for all $t \in [0, T]$, $z \in K$ and $\varepsilon \in [0, \delta]$. This last inequality and the fact that g is locally uniformly Lipschitz, used together with the representation

$$y(t, z, \varepsilon) = \frac{1}{\varepsilon} \int_0^t [f(s, x(s, z, \varepsilon)) - f(s, x(s, z, 0))] ds + \int_0^t g(s, x(s, z, \varepsilon), \varepsilon) ds,$$

imply that

$$\begin{aligned} \|y(t, z_1, \varepsilon) - y(t, z_2, \varepsilon)\| &\leq O(\delta)/\delta \int_0^t \|y(s, z_1, \varepsilon) - y(s, z_2, \varepsilon)\| ds \\ &\quad + O(\delta)/\delta \int_0^t \|x(s, z_1, \varepsilon) - x(s, z_2, \varepsilon)\| ds, \end{aligned}$$

for all $t \in [0, T]$, $z_1, z_2 \in K$ and $\varepsilon \in [0, \delta]$.

We use now the fact that the function $x(t, z, \varepsilon)$ is Lipschitz with respect to z and we deduce

$$\|y(t, z_1, \varepsilon) - y(t, z_2, \varepsilon)\| \leq O(\delta)/\delta \|z_1 - z_2\| + O(\delta)/\delta \int_0^t \|y(s, z_1, \varepsilon) - y(s, z_2, \varepsilon)\| ds.$$

Applying Grönwall lemma (see [12, Lemma 6.2] or [10, Ch. 2, Lemma § 11]) we finally have for all $t \in [0, T]$, $z_1, z_2 \in K$, $\varepsilon \in [0, \delta]$, $\|y(t, z_1, \varepsilon) - y(t, z_2, \varepsilon)\| \leq O(\delta)/\delta \|z_1 - z_2\|$.

(C12) First we note that assumption (A9) of Theorem 2 and the fact that g is locally uniformly Lipschitz with respect to the second variable assure that the following relation holds

$$(4.25) \quad \begin{aligned} &\|g(t, z_1 + \zeta_1, \varepsilon) - g(t, z_1, 0) - g(t, z_2 + \zeta_2, \varepsilon) + g(t, z_2, 0)\| \leq \\ &\leq o(\delta)/\delta \|z_1 - z_2\| + O(\delta)/\delta \|\zeta_1 - \zeta_2\|, \end{aligned}$$

for all $t \in [0, T] \setminus M_\delta$, $z_1, z_2 \in B_\delta(z_0)$, $\varepsilon \in [0, \delta]$ and $\zeta_1, \zeta_2 \in B_\delta(0)$. We introduce the notations $v(t, z, \zeta) = x(t, z + \zeta, 0) - x(t, z, 0)$, $\tilde{\zeta}(s, z, \zeta, \varepsilon) = v(s, z, \zeta) + \varepsilon y(s, z + \zeta, \varepsilon)$ and $u(t, z, \zeta, \varepsilon) = y(t, z + \zeta, \varepsilon) - y(t, z, 0)$. Since the function $x(\cdot, \cdot, 0)$ is C^1 , v is Lipschitz with respect to z on $[0, T] \times K \times B_\delta(0)$ with some constant $o(\delta)/\delta$, we have

$$\begin{aligned} u(t, z, \zeta, \varepsilon) &= y(t, z + \zeta, \varepsilon) - y(t, z, 0) \\ &= \frac{1}{\varepsilon} \int_0^t [f(s, x(s, z + \zeta, \varepsilon)) - f(s, x(s, z + \zeta, 0)) - \varepsilon D_x f(s, x(s, z, 0)) y(s, z, 0)] ds \\ &\quad + \int_0^t [g(s, x(s, z + \zeta, \varepsilon), \varepsilon) - g(s, x(s, z, 0), 0)] ds. \end{aligned}$$

Our aim is to estimate a Lipschitz constant with respect to z of the function u on $[0, T] \times B_\delta(z_0) \times B_\delta(0) \times [0, \delta]$. We will apply Lemma 2, relation (4.25), the fact that g is locally uniformly Lipschitz, and using the following decompositions and estimations that hold for $(s, z, \zeta, \varepsilon) \in [0, T] \times B_\delta(z_0) \times B_\delta(0) \times [0, \delta]$,

$$\begin{aligned} x(s, z + \zeta, \varepsilon) &= x(s, z, 0) + v(s, z, \zeta) + \varepsilon y(s, z, 0) + \varepsilon u(s, z, \zeta, \varepsilon), \\ x(s, z + \zeta, 0) &= x(s, z, 0) + v(s, z, \zeta), \\ x(s, z + \zeta, \varepsilon) &= x(s, z, 0) + \tilde{\zeta}(s, z, \zeta, \varepsilon), \end{aligned}$$

$$\|v(t, z, \zeta)\| \leq o(\delta)/\delta, \quad \|u(t, z, \zeta, \varepsilon)\| \leq o(\delta)/\delta, \quad \|\tilde{\zeta}(s, z, \zeta, \varepsilon)\| \leq \delta O(\delta)/\delta,$$

we obtain

$$\begin{aligned} & \|u(t, z_1, \zeta, \varepsilon) - u(t, z_2, \zeta, \varepsilon)\| \leq \\ & \frac{1}{\varepsilon} \int_0^t [o(\varepsilon) + \varepsilon o(\delta)/\delta] \|x(s, z_1, 0) - x(s, z_2, 0)\| + O(\varepsilon) \|v(s, z_1, \zeta) - v(s, z_2, \zeta)\| + \\ & [o(\varepsilon) + \varepsilon o(\delta)/\delta] \|y(s, z_1, 0) - y(s, z_2, 0)\| + O(\varepsilon) \|u(s, z_1, \zeta, \varepsilon) - u(s, z_2, \zeta, \varepsilon)\| ds + \\ & \int_{(0,t) \setminus M_\delta} o(\delta)/\delta \|x(s, z_1, 0) - x(s, z_2, 0)\| + O(\delta)/\delta \|\tilde{\zeta}(s, z_1, \zeta, \varepsilon) - \tilde{\zeta}(s, z_2, \zeta, \varepsilon)\| ds + \\ & o(\delta)/\delta \|z_1 - z_2\|. \end{aligned}$$

Now we use that some Lipschitz constants with respect to z for the functions x and y on $[0, T] \times B_\delta(z_0) \times [0, \delta]$ are $O(\delta)/\delta$, while for the functions v on $[0, T] \times B_\delta(z_0) \times [0, \delta]$ and $\tilde{\zeta}$ on $[0, T] \times B_\delta(z_0) \times B_\delta(0) \times [0, \delta]$ are $o(\delta)/\delta$, and finally we obtain that

$$\begin{aligned} & \|u(t, z_1, \zeta, \varepsilon) - u(t, z_2, \zeta, \varepsilon)\| \leq \\ & o(\delta)/\delta \|z_1 - z_2\| + O(\delta)/\delta \int_0^t \|u(t, z_1, \zeta, \varepsilon) - u(t, z_2, \zeta, \varepsilon)\| ds. \end{aligned}$$

The conclusion follows after applying the Grönwall inequality. \square

Lemma 4. *We consider the C^1 function Y acting from \mathbb{R}^n into the space of $n \times n$ matrices, the C^2 function $\tilde{P} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $z_* \in \mathbb{R}^n$ such that $\tilde{P}(z_*) = 0$. We denote $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the C^1 function given by $P(z) = Y(z)\tilde{P}(z)$ for all $z \in \mathbb{R}^n$. Then $DP(z_*) = Y(z_*)D\tilde{P}(z_*)$, P is twice differentiable in z_* and, for each $i \in \overline{1, n}$, the Hessian matrix $HP_i(z_*)$ is symmetric.*

Proof. We have $DP(z) = \left(\frac{\partial Y}{\partial z_1}(z)\tilde{P}(z), \dots, \frac{\partial Y}{\partial z_n}(z)\tilde{P}(z) \right) + Y(z)D\tilde{P}(z)$ for all $z \in \mathbb{R}^n$. From this it follows the formula for $DP(z_*)$ since $\tilde{P}(z_*) = 0$.

In order to prove that P is twice differentiable in z_* , taking into account the above expression of DP , it is enough to prove that for each $i \in \overline{1, n}$, $z \mapsto \frac{\partial Y}{\partial z_i}(z)\tilde{P}(z)$ and $z \mapsto Y(z)D\tilde{P}(z)$ are differentiable in z_* . The last map is C^1 , hence it remains to prove the differentiability only for the first one. We fix $i \in \overline{1, n}$. From the relation

$$\begin{aligned} & \frac{\partial Y}{\partial z_i}(z_* + h)\tilde{P}(z_* + h) - \frac{\partial Y}{\partial z_i}(z_*)\tilde{P}(z_*) = \\ & \frac{\partial Y}{\partial z_i}(z_* + h) \left(\tilde{P}(z_* + h) - \tilde{P}(z_*) \right) = \frac{\partial Y}{\partial z_i}(z_* + h)D\tilde{P}(z_*) + o(h), \end{aligned}$$

we deduce that $z \mapsto \frac{\partial Y}{\partial z_i}(z)\tilde{P}(z)$ is differentiable in z_* and that

$$D \left(\frac{\partial Y}{\partial z_i} \cdot \tilde{P} \right) (z_*) = \frac{\partial Y}{\partial z_i}(z_*)D\tilde{P}(z_*).$$

In order to prove that the Hessian matrix $HP_i(z_*)$ is symmetric, for every $j, k \in \{1, \dots, n\}$ we must prove that

$$\frac{\partial^2 P_i}{\partial z_j \partial z_k}(z_*) = \frac{\partial^2 P_i}{\partial z_k \partial z_j}(z_*).$$

We denote by $Y_i(z)$ the i -th row of the $n \times n$ matrix $Y(z)$. For all $z \in \mathbb{R}^n$ we have

$$\frac{\partial P_i}{\partial z_j}(z) = Y_i(z) \frac{\partial \tilde{P}}{\partial z_j}(z) + \frac{\partial Y_i}{\partial z_j}(z) \tilde{P}(z).$$

Then

$$\frac{\partial^2 P_i}{\partial z_j \partial z_k}(z_*) = \frac{\partial Y_i}{\partial z_k}(z_*) \frac{\partial \tilde{P}}{\partial z_j}(z_*) + Y_i(z_*) \frac{\partial^2 \tilde{P}}{\partial z_j \partial z_k}(z_*) + \frac{\partial Y_i}{\partial z_j}(z_*) \frac{\partial \tilde{P}}{\partial z_k}(z_*).$$

Since \tilde{P} is C^2 it is easy to check the symmetry of this last relation with respect to (j, k) . \square

Proof of Theorem 2. We need to study the zeros of the function $z \mapsto x(T, z, \varepsilon) - z$, or equivalently of

$$F(z, \varepsilon) = Y^{-1}(T, z)(x(T, z, \varepsilon) - z).$$

The function F is well defined at least for any z in some small neighborhood of \mathcal{Z} and any $\varepsilon \geq 0$ sufficiently small. We will apply Theorem 1. We denote

$$P(z) = Y^{-1}(T, z)(x(T, z, 0) - z), \quad Q(z, \varepsilon) = Y^{-1}(T, z)y(T, z, \varepsilon),$$

where $y(t, z, \varepsilon) = [x(t, z, \varepsilon) - x(t, z, 0)]/\varepsilon$, like in Lemma 3. Hence $F(z, \varepsilon) = P(z) + \varepsilon Q(z, \varepsilon)$.

The fact that f is C^2 assures that the function $z \mapsto x(T, z, 0)$ is also C^2 (see [24, Ch. 4, § 24]). Since (see [10, Ch. III, Lemma § 12]) $(Y^{-1}(\cdot, z))^*$ is a fundamental matrix solution of the system

$$\dot{u} = -(D_x f(t, x(t, z, 0), 0))^* u,$$

and f is C^2 , we have that the matrix function $(t, z) \mapsto (Y^{-1}(t, z))^*$ is C^1 . Therefore the matrix function $(t, z) \mapsto Y^{-1}(t, z)$, and consequently also the function P are C^1 .

By Lemma 3 we now conclude that Q is continuous, locally uniformly Lipschitz with respect to z , and

$$(4.26) \quad Q(z, 0) = \int_0^T Y^{-1}(s, z) g(s, x(s, z, 0), 0) ds.$$

Since, by our hypothesis (A6), $x(\cdot, z, 0)$ is T -periodic for all $z \in \mathcal{Z}$ we have that $x(T, z, 0) - z = 0$ for all $z \in \mathcal{Z}$, and consequently $P(z) = 0$ for all $z \in \mathcal{Z}$. This means that hypothesis (A1) of Theorem 1 holds. Moreover applying Lemma 4 we have that

$$DP(z) = Y^{-1}(T, z) \left(\frac{\partial x}{\partial z}(T, z, 0) - I_{n \times n} \right) \quad \text{for any } z \in \mathcal{Z},$$

and P satisfies hypothesis (A3) of Theorem 1. But $(\partial x / \partial z)(\cdot, z, 0)$ is the normalized fundamental matrix of the linearized system (4.21) (see [17, Theorem 2.1]). Therefore $(\partial x / \partial z)(t, z, 0) = Y(t, z)Y^{-1}(0, z)$, and we can write

$$(4.27) \quad DP(z) = Y^{-1}(0, z) - Y^{-1}(T, z) \quad \text{for any } z \in \mathcal{Z}.$$

Using our hypothesis (A7) we see that also assumption (A2) of Theorem 1 is satisfied. From the definition of G and relation (4.26) we have that

$$G(\alpha) = \pi Q \left(S \begin{pmatrix} \alpha \\ \beta_0(\alpha) \end{pmatrix}, 0 \right).$$

That is, the function denoted in Theorem 1 by \hat{Q} is here G , and it satisfies the hypotheses of Theorem 1. Moreover, note that when G satisfies (A8) then assumption (A4) of Theorem 1 is fulfilled.

(C7) Follows from (C1) of Theorem 1.

(C8) Follows from (C2) of Theorem 1.

(C9) In order to prove the uniqueness of the T -periodic solution, it remains only to check (A5) of Theorem 1. For doing this we show that the function $(z, \zeta, \varepsilon) \in B_\delta(z_0) \times B_\delta(0) \times [0, \delta] \mapsto Q(z + \zeta, \varepsilon) - Q(z, 0)$ is Lipschitz with respect to z with some constant $o(\delta)/\delta$. We write

$$\begin{aligned} Q(z + \zeta, \varepsilon) - Q(z, 0) &= Y^{-1}(T, z + \zeta) [y(T, z + \zeta, \varepsilon) - y(T, z, 0)] + \\ &\quad [Y^{-1}(T, z + \zeta) - Y^{-1}(T, z)] y(T, z, 0). \end{aligned}$$

It is known that for proving that a sum of two functions is Lipschitz with some constant $o(\delta)/\delta$, it is enough to prove that each function is Lipschitz with such constant; while in order to prove that a product of two functions is Lipschitz with some constant $o(\delta)/\delta$, it is sufficient to prove that both functions are Lipschitz and only one of them is bounded by some constant $o(\delta)/\delta$ and Lipschitz with respect to z with some constant $o(\delta)/\delta$.

By Lemma 3 we know that the function $z \in B_\delta(z_0) \mapsto y(T, z, 0)$ is Lipschitz. The fact that $z \mapsto Y^{-1}(T, z)$ is C^1 assures that $(z, \zeta) \in B_\delta(z_0) \times B_\delta(0) \mapsto Y^{-1}(T, z + \zeta)$ is Lipschitz with respect to z .

From Lemma 3 we have that the function

$$(z, \zeta, \varepsilon) \in B_\delta(z_0) \times B_\delta(0) \times [0, \delta] \mapsto y(T, z + \zeta, \varepsilon) - y(T, z, 0)$$

is bounded by some constant $o(\delta)/\delta$ and Lipschitz with some constant $o(\delta)/\delta$. Since $z \mapsto Y^{-1}(T, z)$ is C^1 , the same is true for the function

$$(z, \zeta) \in B_\delta(z_0) \times B_\delta(0) \mapsto [Y^{-1}(T, z + \zeta) - Y^{-1}(T, z)].$$

Hence Q satisfies (A5) of Theorem 1 and the conclusion holds. □

By using Theorem 2 we can provide now a result which includes both the existence-uniqueness results by Malkin [21] and by Melnikov [?]. The main contribution of our result is that we do not impose any assumptions on the algebraic multiplicity of the multiplier $+1$ of (1.2). Also, the condition imposed on the function $z \mapsto g(t, z, \varepsilon)$ is weaker than the C^1 condition used by Malkin and Melnikov. In particular, our Theorem 3 covers the class of piecewise differentiable systems.

Theorem 3. *Assume that $f \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C^0(\mathbb{R} \times \mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ are T -periodic in the first variable, and that g is locally uniformly Lipschitz with respect to the second variable. Assume that the unperturbed system (4.20) satisfies the following conditions.*

- (A10) *There exists an open ball $U \subset \mathbb{R}^k$ with $k \leq n$ and a function $\xi \in C^1(\overline{U}, \mathbb{R}^n)$ such that for any $h \in \overline{U}$ the $n \times k$ matrix $D\xi(h)$ has rank k and $\xi(h)$ is the initial condition of a T -periodic solution of (4.20). ****
- (A11) *For each $h \in \overline{U}$ the linear system (4.21) with $z = \xi(h)$ has the Floquet multiplier $+1$ with the geometric multiplicity equal to k .*

Let $u_1(\cdot, h), \dots, u_k(\cdot, h)$ be linearly independent T -periodic solutions of the adjoint linear system

$$(4.28) \quad \dot{u} = -(D_x f(t, x(t, \xi(h), 0)))^* u,$$

*such that $u_1(0, h), \dots, u_k(0, h)$ are C^1 with respect to h *** and define the function $M : \overline{U} \rightarrow \mathbb{R}^k$ (called the Malkin's bifurcation function) by*

$$M(h) = \int_0^T \begin{pmatrix} \langle u_1(s, h), g(s, x(s, \xi(h), 0), 0) \rangle \\ \dots \\ \langle u_k(s, h), g(s, x(s, \xi(h), 0), 0) \rangle \end{pmatrix} ds.$$

Then the following statements hold.

- (C13) *For any sequences $(\varphi_m)_{m \geq 1}$ from $C^0(\mathbb{R}, \mathbb{R}^n)$ and $(\varepsilon_m)_{m \geq 1}$ from $[0, 1]$ such that $\varphi_m(0) \rightarrow \xi(h_0) \in \xi(\overline{U})$, $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$ and φ_m is a T -periodic solution of (4.19) with $\varepsilon = \varepsilon_m$, we have that $M(h_0) = 0$.*
- (C14) *If $M(h) \neq 0$ for any $h \in \partial U$ and $d(M, U) \neq 0$, then there exists $\varepsilon_1 > 0$ sufficiently small such that for each $\varepsilon \in (0, \varepsilon_1]$ there is at least one T -periodic solution φ_ε of system (4.19) such that $\rho(\varphi_\varepsilon(0), \xi(\overline{U})) \rightarrow 0$ as $\varepsilon \rightarrow 0$.*

In addition we assume that there exists $h_0 \in U$ such that $M(h_0) = 0$, $M(h) \neq 0$ for all $h \in \overline{U} \setminus \{h_0\}$ and $d(M, U) \neq 0$. Moreover we assume that hypothesis (A9) of Theorem 2 holds with $z_0 = \xi(h_0)$ and that

- (A12) *There exists $\delta_1 > 0$ and $L_M > 0$ such that*

$$||M(h_1) - M(h_2)|| \geq L_M ||h_1 - h_2||, \text{ for all } h_1, h_2 \in B_{\delta_1}(h_0).$$

Then the following conclusion holds.

- (C15) *There exists $\delta_2 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, φ_ε is the only T -periodic solution of (4.19) with initial condition in $B_{\delta_2}(z_0)$. Moreover $\varphi_\varepsilon(0) \rightarrow \xi(h_0)$ as $\varepsilon \rightarrow 0$.*

*****Remark.** *The existence of k linearly independent T -periodic solutions of the adjoint linear system (4.28) follows by hypothesis (A10) (see e.g. [10, Ch. III, § 23, Theorem 2]). Indeed, we have that $y_i(t, h) = D_z x(t, \xi(h), 0) D_{h_i} \xi(h)$ for $i \in \overline{1, k}$ are solutions of (4.21) and they are linearly independent on the base of (A10). The assertion follows by the fact that a linear system and its adjoint have the same number of linearly independent solutions. Moreover, hypothesis (A11) assures that there is no other T -periodic solution to (4.21) linearly independent of these. ****

*****Proof.** We apply Theorem 2. For the moment we describe the set \mathcal{Z} that appear in hypothesis (A6) as $\mathcal{Z} = \bigcup_{h \in \overline{U}} \{\xi(h)\}$. First we find the matrix S such that hypothesis (A7) holds. In order to achieve this, for each $z \in \mathcal{Z}$ we denote $U(t, z)$ some fundamental matrix solution of (4.28) that has in its first k columns the T -periodic solutions u_1, \dots, u_k and such that $U(0, z)$ is C^1 . Then the first k columns of the matrix $U(0, z) - U(T, z)$ are null vectors. The matrix $Y(t, z)$ such that $Y^{-1}(t, z) = [U(t, z)]^*$ is a fundamental matrix solution of (4.21), i.e. of the system ($z = \xi(h) \in \mathcal{Z}$)

$$(4.29) \quad \dot{y} = D_x f(t, x(t, \xi(h), 0)) y.$$

Then the first k lines of the matrix $Y(0, z)^{-1} - Y(T, z)^{-1}$ are null vectors. Since the Floquet multiplier 1 of (4.21) has geometric multiplicity k we have that the matrix $Y^{-1}(0, z) - Y^{-1}(T, z)$ has range $n - k$. Hence this matrix has $n - k$ linearly independent columns. We claim that there exists an invertible matrix S such that the matrix $(Y^{-1}(0, z) - Y^{-1}(T, z)) S$ has in the first k lines null vectors and in the lower right corner some $(n - k) \times (n - k)$ invertible matrix $\Delta(z)$. With this we prove that (A7) holds. In order to justify the claim we note first that whatever the matrix S would be, the first k lines of $(Y^{-1}(0, z) - Y^{-1}(T, z)) S$ are null vectors. Now we choose an invertible matrix S such that its last $(n - k)$ columns are vectors of the form

$$e_i = \begin{pmatrix} 0_{(i-1) \times 1} \\ 1 \\ 0_{(n-i) \times 1} \end{pmatrix}, \quad i \in \overline{1, n}$$

distributed in such a way that the $n - k$ linearly independent columns of $Y^{-1}(0, z) - Y^{-1}(T, z)$ become the last $n - k$ columns of $(Y^{-1}(0, z) - Y^{-1}(T, z)) S$. Now it is easy to see that the $(n - k) \times (n - k)$ matrix from the lower right corner of $(Y^{-1}(0, z) - Y^{-1}(T, z)) S$ is invertible.

Now we come back to prove (A6). By taking the derivative with respect to $h \in U$ of $\dot{x}(t, \xi(h)) = f(t, x(t, \xi(h)))$ we obtain that $D_\xi x(\cdot, \xi(h)) \cdot D\xi(h)$ is a matrix solution for (4.29). But $x(\cdot, \xi(h))$ is T -periodic for any $h \in U$, therefore $D_\xi x(\cdot, \xi(h)) \cdot D\xi(h)$ is T -periodic. This fact assures that each column of $D\xi(h)$ is the initial condition of some T -periodic solution of (4.29) and these T periodic solutions are the columns of $Y(t, \xi(h)) Y^{-1}(0, \xi(h)) D\xi(h)$. Then $Y(T, \xi(h)) Y^{-1}(0, \xi(h)) D\xi(h) = D\xi(h)$, that further gives $[Y^{-1}(0, \xi(h)) - Y^{-1}(T, \xi(h))] S S^{-1} D\xi(h) = 0$. Hence the columns of $S^{-1} D\xi(h)$ belong to the kernel of $[Y^{-1}(0, \xi(h)) - Y^{-1}(T, \xi(h))] S$. Since

(A7) holds we have that the kernel of $[Y^{-1}(0, \xi(h)) - Y^{-1}(T, \xi(h))] S$ contains vectors whose last $n - k$ components are null. We deduce that there exists some $k \times k$ matrix, denoted by Ψ , such that

$$(4.30) \quad S^{-1}D\xi(h) = \begin{pmatrix} \Psi \\ 0_{(n-k) \times k} \end{pmatrix}.$$

Since by the assumption (A10) the matrix $D\xi(h)$ has rank k and S^{-1} is invertible, we have that the matrix $S^{-1}D\xi(h)$ should also have rank k , that is only possible if

$$(4.31) \quad \det \Psi \neq 0.$$

We fix some $h_* \in U$ and we denote $\alpha_* = \pi S^{-1}\xi(h_*)$. Using (4.30) and (4.31), and applying the Implicit Function Theorem we have that there exists an open ball, neighborhood of α_* , denoted $V \subset \mathbb{R}^k$, and a C^1 function $\tilde{h} : \bar{V} \rightarrow U$ such that

$$(4.32) \quad \pi S^{-1}\xi(\tilde{h}(\alpha)) = \alpha \quad \text{for any } \alpha \in \bar{V}.$$

Now we define the C^1 function $\beta_0 : \bar{V} \rightarrow \mathbb{R}^{n-k}$ as $\beta_0(\alpha) = \pi^\perp S^{-1}\xi(\tilde{h}(\alpha))$. Note that $S \begin{pmatrix} \alpha \\ \beta_0(\alpha) \end{pmatrix} = \xi(\tilde{h}(\alpha))$. Hence the assumption (A6) of Theorem 2 is satisfied with S , V and β_0 defined as above.

The bifurcation function G defined in Theorem 2 can be written using our notations as

$$G(\alpha) = \pi \int_0^T Y^{-1}(s, \xi(\tilde{h}(\alpha))) g(s, x(s, \xi(\tilde{h}(\alpha))), 0) ds.$$

Since $Y^{-1}(s, \xi(h)) = [U(t, h)]^*$ (see the beginning of the proof) we have that in the first k lines of $Y^{-1}(s, \xi(h))$ are the vectors $(u_1(s, h))^*, \dots, (u_n(s, h))^*$ and so

$$G(\alpha) = \int_0^T \begin{pmatrix} \langle u_1(s, \tilde{h}(\alpha)), g(s, x(s, \xi(\tilde{h}(\alpha))), 0), 0 \rangle \\ \dots \\ \langle u_k(s, \tilde{h}(\alpha)), g(s, x(s, \xi(\tilde{h}(\alpha))), 0), 0 \rangle \end{pmatrix} ds.$$

From here one can see that there is the following relation between G and the Malkin bifurcation function M ,

$$(4.33) \quad G(\alpha) = M(\tilde{h}(\alpha)) \quad \text{for any } \alpha \in \bar{V}.$$

(C13) Follows from (C7) of Theorem 2.

(C14) Without loss of generality (we can diminish U , if necessary) we can consider that \tilde{h} is a homeomorphism from V onto U and taking into account that C is an invertible matrix by [18, Theorem 26.4] we have

$$\deg(G, V) = \deg(M, U).$$

Thus (C14) follows applying conclusion (C8) of Theorem 2.

(C15) We need only to prove assumption (A8) of Theorem 2 provided that our hypothesis (A12) holds. First, taking the derivative of (4.32) with respect to α and using (4.30), we obtain that $D\tilde{h}(\alpha_*) = \Psi^{-1}$, hence it is invertible and, moreover, $L_h = \|D\tilde{h}(\alpha_*)\|/2 \neq 0$. We have that there exists $\delta > 0$ sufficiently small such that $\|D\tilde{h}(\alpha) - D\tilde{h}(\alpha_*)\| \leq L_h$ for all $\alpha \in B_\delta(\alpha_*)$. Using the generalized Mean Value Theorem (see [9, Proposition 2.6.5]), we have that

$$\|\tilde{h}(\alpha_1) - \tilde{h}(\alpha_2)\| \geq L_h \|\alpha_1 - \alpha_2\| \text{ for all } \alpha_1, \alpha_2 \in B_\delta(\alpha_0).$$

Since C is invertible, M satisfies (A10) and (4.33), we deduce that G satisfies hypothesis (A8) of Theorem 2. □

5. AN EXAMPLE

In this section we illustrate Theorem 3 by studying the existence of 2π -periodic solutions for the following four-dimensional nonsmooth system

$$(5.34) \quad \begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1), \\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2 - 1) + \varepsilon(\sin t + \varphi(k_1 x_3)), \\ \dot{x}_3 &= x_4 - x_3(x_3^2 + x_4^2 - 1), \\ \dot{x}_4 &= -x_3 - x_4(x_3^2 + x_4^2 - 1) + \varepsilon\varphi(k_2 x_2), \end{aligned}$$

when $\varepsilon > 0$ is sufficiently small, k_1 and k_2 are arbitrary reals and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is the piecewise linear function

$$\varphi(x) = \begin{cases} -1, & \text{for } x \in (-\infty, -1), \\ x, & \text{for } x \in [-1, 1], \\ 1, & \text{for } x \in (1, \infty). \end{cases}$$

Before proceeding to this study we write down a sufficient condition in order that the function g satisfies (A9). This is of general interest, not only for this example. It was also used in [6].

(A13) For any $\delta > 0$ sufficiently small there exists $M_\delta \subset [0, T]$ Lebesgue measurable with $\text{mes}(M_\delta) = o(\delta)/\delta$ and such that for every $t \in [0, T] \setminus M_\delta$ and for all $z \in B_\delta(z_0)$, $\varepsilon \in [0, \delta]$, $\|D_z g(t, z, \varepsilon) - D_z g(t, z_0, 0)\| \leq o(\delta)/\delta$.

The fact that (A13) implies (A9) follows from the Mean Value Theorem.

Coming back to system (5.34) we define $g : \mathbb{R} \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$ as $g(t, z_1, z_2, z_3, z_4) = (0, \sin t + \varphi(k_1 z_3), 0, \varphi(k_2 z_2))$. This function is uniformly locally Lipschitz and satisfies (A13) (hence also (A9)) for any $z_0 = (z_1^0, z_2^0, z_3^0, z_4^0)$ with $|k_2 z_2^0| \neq 1$ and $|k_1 z_3^0| \neq 1$.

We study now the unperturbed system. For $\varepsilon = 0$ system (5.34) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1), & \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2 - 1), \\ \dot{x}_3 &= x_4 - x_3(x_3^2 + x_4^2 - 1), & \dot{x}_4 &= -x_3 - x_4(x_3^2 + x_4^2 - 1), \end{aligned}$$

and it has a family of 2π -periodic orbits, whose initial conditions are given by

$$\xi(\theta, \eta) = (\sin \theta, \cos \theta, \sin \eta, \cos \eta), \quad (\theta, \eta) \in \mathbb{R}^2.$$

We have $x(t, \xi(\theta, \eta), 0) = (\sin(t + \theta), \cos(t + \theta), \sin(t + \eta), \cos(t + \eta))$.

It is easy to see that the function $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ satisfies assumption (A10) of Theorem 3. We consider now the linearized system,

$$(5.35) \quad \dot{y} = D_x f(t, x(t, \xi(\theta, \eta), 0))y,$$

where $f(x_1, x_2, x_3, x_4) = (x_2 - x_1(x_1^2 + x_2^2 - 1), -x_1 - x_2(x_1^2 + x_2^2 - 1), x_4 - x_3(x_3^2 + x_4^2 - 1), -x_3 - x_4(x_3^2 + x_4^2 - 1))$, and the following 2π -periodic matrix

$$\Phi(t, (\theta, \eta)) = \begin{pmatrix} -\cos(t + \theta) & 0 & \sin(t + \theta) & 0 \\ \sin(t + \theta) & 0 & \cos(t + \theta) & 0 \\ 0 & -\cos(t + \eta) & 0 & \sin(t + \eta) \\ 0 & \sin(t + \eta) & 0 & -\cos(t + \eta) \end{pmatrix}.$$

Denoting $\hat{\Phi}(t, (\theta, \eta)) = \Phi(t, (\theta, \eta))\Phi^{-1}(0, (\theta, \eta))$, it can be checked that the normalized fundamental matrix of system (5.35) is

$$\hat{Y}(t, \xi(\theta, \eta)) = \hat{\Phi}(t, (\theta, \eta)) \begin{pmatrix} I_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & e^{-2t} I_{2 \times 2} \end{pmatrix},$$

and that it satisfies the hypothesis (A11) of Theorem 3.

A couple of linearly independent 2π -periodic solutions of the adjoint system to system (5.35) is

$$\begin{aligned} u_1(t) &= (-\cos(t + \theta), \sin(t + \theta), 0, 0)^T, \\ u_2(t) &= (0, 0, -\cos(t + \eta), \sin(t + \eta))^T. \end{aligned}$$

Therefore the Malkin's bifurcation function $M(\theta, \eta)$ takes the form

$$M(\theta, \eta) = \begin{pmatrix} M_1(\theta, \eta) \\ M_2(\theta, \eta) \end{pmatrix} = \begin{pmatrix} \int_0^{2\pi} \sin(s + \theta) [\sin s + \varphi(k_1 \sin(s + \eta))] ds \\ \int_0^{2\pi} \sin(s + \eta) \varphi(k_2 \cos(s + \theta)) ds \end{pmatrix}.$$

We denote

$$\begin{aligned} I(k) &= \int_0^{2\pi} \sin t \varphi(k \sin t) dt, \quad i_1 = I(k_1), \\ J(k) &= \int_0^{2\pi} \cos t \varphi(k \cos t) dt, \quad j_2 = J(k_2). \end{aligned}$$

Then

$$\begin{aligned} M_1(\theta, \eta) &= \pi \cos \theta + i_1 \cos(\theta - \eta), \\ M_2(\theta, \eta) &= -j_2 \sin(\theta - \eta). \end{aligned}$$

It is easy to see that the function M is differentiable and that the determinant of its Jacobian at each point (θ, η) is

$$\det(DM(\theta, \eta)) = -\pi j_2 \sin \theta \cos(\theta - \eta).$$

Studying the zeros of the function M , we obtain the following results.

If $j_2 \neq 0$ and $|i_1| < \pi$, M has exactly 4 zeros, namely

$$(a_1, \pi + a_1), \quad (\pi + a_1, a_1), \quad (\pi - a_1, \pi - a_1), \quad (2\pi - a_1, 2\pi - a_1),$$

and all have nonvanishing index (here $a_1 = \arccos(i_1/\pi)$).

If $j_2 = 0$, we have that $M_2 \equiv 0$, hence the zeros are not isolated.

If $j_2 \neq 0$ and $|i_1| = \pi$, M has exactly 2 zeros, namely

$$(a_1, \pi + a_1), \quad (\pi - a_1, \pi - a_1),$$

and both have index 0 (note that a_1 is 0 or π).

If $j_2 \neq 0$ and $|i_1| > \pi$, M has no zeros.

In order to complete the study, we calculate

$$I(k) = J(k) = \begin{cases} k\pi, & 0 \leq k \leq 1, \\ 2k \arcsin \frac{1}{k} + 2\frac{\sqrt{k^2-1}}{k}, & k > 1. \end{cases}$$

We note that I and J are odd functions. For $k > 1$, $I'(k) = \arcsin \frac{1}{k} - \frac{1}{k}\sqrt{1 - \frac{1}{k^2}} > \frac{1}{k} - \frac{1}{k}\sqrt{1 - \frac{1}{k^2}} > 0$. Then $I(k) > I(1) = \pi$ for all $k > 1$. In fact $|I(k)| > \pi$ if and only if $|k| > 1$. Also $|I(k)| < \pi$ if and only if $|k| < 1$, $|I(k)| = \pi$ if and only if $|k| = 1$, and $J(k) = 0$ if and only if $k = 0$.

One can see that when $k_2 \neq 0$ and $|k_1| < 1$ the bifurcation function M has exactly four zeros. The values of the function ξ on these zeros are of the form $(\pm \sin a_1, \pm \cos a_1, \pm \sin a_1, \pm \cos a_1)$ where $a_1 = \arccos k_1$.

Using all these facts and applying Theorem 3, we obtain the following result concerning the existence of 2π -solutions of system (5.34) when $k_2 \neq 0$.

Proposition 1. *Let $k_2 \neq 0$ and $\varepsilon > 0$ be sufficiently small.*

If $|k_1| > 1$ system (5.34) has no 2π -periodic solutions with initial conditions converging to some point of $\xi(\mathbb{R} \times \mathbb{R})$ as $\varepsilon \rightarrow 0$.

If $|k_1| < 1$ system (5.34) has at least four 2π -periodic solutions with initial conditions converging to exactly four points of $\xi(\mathbb{R} \times \mathbb{R})$ as $\varepsilon \rightarrow 0$. If $|k_1 k_2| \neq 1$ then the obtained 2π -periodic solutions are isolated.

For $k_2 = 0$ the behavior of system (5.34) is the same as of the uncoupled system whose last two equations remain the same and the first two change to

$$(5.36) \quad \begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1), \\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2 - 1) + \varepsilon(\sin t + \varphi(k_1 \sin(t + \eta))). \end{aligned}$$

In the following we study the existence of 2π -periodic solutions of the above planar system. We consider only the case that $(k_1, \eta) \in \mathbb{R} \times \mathbb{R} \setminus \{(1, \pi), (-1, 0)\}$, otherwise the perturbation term is identically zero. Note also that the perturbation term

does not depend on the space variable. System (5.36) with $\varepsilon = 0$ has a family of 2π -periodic orbits whose initial conditions are given by

$$\xi(\theta) = (\sin \theta, \cos \theta), \quad \theta \in \mathbb{R}.$$

For each fixed η the Malkin's bifurcation function is

$$M^\eta(\theta) = \pi \cos \theta + i_1 \cos(\theta - \eta),$$

and it can be easily seen that it has exactly 2 zeros, and both have nonvanishing index. If $k_1 = 0$ or $\eta \in \{0, \pi\}$, the zeros of M^η are $\pi/2$ and $3\pi/2$. Otherwise the zeros are $\pi - \theta^*$ and $2\pi - \theta^*$, where $\theta^* = \arctan \frac{\pi + i_1 \cos \eta}{i_1 \sin \eta}$.

Hence applying Theorem 3 we obtain the result.

Proposition 2. *Let $(k_1, \eta) \in \mathbb{R} \times \mathbb{R} \setminus \{(1, \pi), (-1, 0)\}$ and $\varepsilon > 0$ be sufficiently small. Then system (5.36) has two isolated 2π -periodic solutions with initial conditions converging to exactly two points of $\xi^\eta(\mathbb{R})$ as $\varepsilon \rightarrow 0$.*

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